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Lattice Gas Model in Random Medium and Open Boundaries: Hydrodynamic and Relaxation to the Steady State

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Abstract We consider a lattice gas interacting by the exclusion rule in the presence of a random field given by i.i.d. bounded random variables in a bounded domain in contact with particles reservoir at different densities. We show, in dimensions $d \geq 3$, that the rescaled empirical density field almost surely, with respect to the random field, converges to the unique weak solution of a quasilinear parabolic equation having the diffusion matrix determined by the statistical properties of the external random field and boundary conditions determined by the density of the reservoir. Further we show that the rescaled empirical density field, in the stationary regime, almost surely with respect to the random field, converges to the solution of the associated stationary transport equation.

Keywords Random environment · Nongradient systems · Stationary nonequilibrium states

1 Introduction

In the last years there has been several papers devoted in understanding macroscopic properties of non equilibrium systems. Typical examples are systems in contact with two thermostats at different temperature or with two reservoirs at different densities. A mathematical model of open boundary systems is provided by stochastic models of interacting particles systems performing a local reversible dynamics (for example a reversible hopping dynamics) in a domain and some external mechanism of creation and annihilation of particles on

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the boundary of the domain, modeling the reservoirs, which makes the full process non reversible. There has been important classes of models, see for example [4, 6, 7, 14, 22] in which it has been proved the law of large numbers for the empirical density in the stationary regime. Typical generic feature of these systems is that they exhibit long range correlation in their steady state. More recently breakthroughs were achieved analyzing the large deviations principle for the stationary measure. We refer to [2, 3] for a review of works on the statistical mechanics of non equilibrium processes based on the analysis of large deviations properties of microscopic systems.

In this paper we consider a particles system evolving according to local conservative dynamics (Kawasaki) with hard core exclusion rule and with rates depending on a quenched random field in a cylinder domain $d \geq 3$ in which the basis, denoted Γ, are kept at different densities. The rates of the interaction are chosen so that the system satisfies a detailed balance condition with respect to a family of random Bernoulli measures (the random field Ising model at infinite temperature). To model the presence of the reservoirs, we superimpose at the boundary, to the local-conservative dynamics, a birth and death process. The rates of this birth and death process depend on the realizations of the random field and are chosen so that a random Bernoulli measure with a suitable choice of the chemical potential is reversible for it. This latter dynamic is of course not conservative and keeps the fixed value of the density at the boundary. There is a flow of density through the full system and the full dynamic is not reversible.

We derive for such a model the hydrodynamic limit dealing simultaneously both with the randomness of the rates and with the open boundaries conditions. The rescaled empirical density field almost surely, with respect to the random field, converges to the unique weak solution of the quasilinear parabolic equation (2.15). In addition to this we prove the hydrostatics, i.e. the rescaled empirical density field almost surely, with respect to the random field, converges under the unique stationary measure of the evolution process to the stationary solution of (2.15). This is obtained deriving first the hydrodynamic for the empirical density field distributed according to the stationary measure. Then we exploit that the stationary solution of (2.15) is unique and is a global attractor for the macroscopic evolution. These two ingredients, together with the weak compactness of the space of measures allow to conclude. Similar strategy for proving the hydrostatic is used in the paper by Farfan Vargas, Landim and Mourragui [9].

The bulk dynamic models electron transport in doped crystals. In this case the exclusion rule is given by the Pauli principle and the presence of impurities in the crystals is the origin of the presence of quenched random field, see [12]. The transport properties of such systems in the case of periodic boundary condition on Γ has been studied by Faggionato and Martinelli [8]. They derived in $d \geq 3$, the hydrodynamic limit and gave a variational formula for the bulk diffusion. Later, Quastel [20] derived in all dimensions for the same model investigated by [8] the hydrodynamic limit for the local empirical density and proved some regularity properties for the bulk diffusion, see for further comments Sect 2.2.

Applying the method proposed by Quastel, we could extend our results in all dimensions. Since our aim is to understand the role of the randomness in the non stationary and stationary regime and not the role of dimensions in the bulk dynamics we state and prove our results in $d \geq 3$. Dynamical Large deviations for the same model and always with periodic boundary conditions have been derived in [19] as special case of a more general system discussed there. The bulk dynamics is of the so-called *nongradient* type. Roughly speaking, the gradient condition says that the microscopic current is already the gradient of a function of the density field. Further it is not translation invariant, for a given disorder configuration. To prove the hydrodynamic behavior of the system, we follow the entropy method introduced by Guo, Papanicolaou and Varadhan [11] together with the results of [8]. The entropy method relies on an estimate of the entropy of the states of process with respect to a reference invariant state. By the general theory of Markov Processes the entropy of the state of a process with respect to an invariant state decreases in time. The main problem is that in the model considered the reference invariant state is not explicitly known. To overcome this difficulty we compute the entropy of the state of the process with respect to a product measure with slowly varying profile. Since this measure is not invariant, the entropy does not need to decrease and we need to estimate the rate at which it increases. This type of strategy has been used in previous papers dealing with the same type of problems, see [14, 17], which considered generalized exclusion process of non gradient type. The main difference with the previous mentioned papers is the presence of the randomness in the model considered here. This forces to consider on the boundary jump processes with rates depending on the external random field. Important step to derive the final result is then a convenient application of the ergodic theorem, see Proposition 3.4.

2 The Model and the Main Results

2.1 The Model

We consider the d-dimensional lattice $\mathbb{Z}^d$ with sites $x=(x_1,\dots,x_d)$ and canonical basis $\mathcal{E}=\{e_1,\dots,e_d\}$ and we assume in all the paper that $d\ge 3$. We denote by $\Lambda:=[-1,1]\times \mathbf{T}^{d-1}$, where $\mathbf{T}^{d-1}$ is the $(d-1)$-dimensional torus of diameter 1 and by Γ the boundary of Λ.

Fix an integer $N\ge 1$. Denote by $\Lambda_N\equiv\{-N,\dots,N\}\times\mathbf{T}_N^{d-1}$ the cylinder in $\mathbb{Z}^d$ of length $2N+1$ with basis the $(d-1)$-dimensional discrete torus $\mathbf{T}_N^{d-1}$ and by $\Gamma_N=\{x\in\Lambda_N\,|\,x_1=\pm N\}$ the boundary of Λ_N. The elements of Λ_N will be denoted by letters $x,y,\dots$ and the elements of Λ by $u,v,\dots$.

The disorder configuration is stochastically chosen by a translational invariant product measure $\mathbb{P}$ on $\Sigma_D=[-A,A]^{\mathbb{Z}^d}$, where A is a fixed positive number. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$, and by $\alpha\equiv\{\alpha(x),\ x\in\mathbb{Z}^d\}$, $\alpha(x)\in[-A,A]$, a disorder configuration in Σ_D. A configuration $\alpha\in\Sigma_D$ induces in a natural way a disorder configuration α_N on Λ_N, by identifying a cube centered at the origin of side $2N+1$ with Λ_N. By a slight abuse of notation whenever in the following we refer to a disorder configuration either on Λ_N or on $\mathbb{Z}^d$ we denote it by α. We denote by $\mathcal{S}_N\equiv\{0,1\}^{\Lambda_N}$ and $\mathcal{S}\equiv\{0,1\}^{\mathbb{Z}^d}$ the configuration spaces, both equipped with the product topology; elements of $\mathcal{S}_N$ or $\mathcal{S}$ are denoted by η, so that $\eta(x)=1$, resp 0, if the site x is occupied, resp empty, for the configuration η. Given $\alpha\in\Sigma_D$, we consider the random Hamiltonian $H^\alpha:\mathcal{S}_N\to\mathbb{R}$,

$$H^\alpha(\eta)=-\sum_{x\in\Lambda_N}\alpha(x)\eta(x). \tag{2.1}$$

We denote by $\mu_N^{\alpha,\lambda}$ the grand canonical random Gibbs measure on $\mathcal{S}_N$ associated to the Hamiltonian (2.1) with chemical potential $\lambda\in\mathbb{R}$, i.e the random Bernoulli product measure

$$\mu_N^{\alpha,\lambda}(\eta)=\prod_{x\in\Lambda_N}\left\{\frac{e^{[\alpha(x)+\lambda]\eta(x)}}{e^{[\alpha(x)+\lambda]}+1}\right\}. \tag{2.2}$$

When $\lambda=0$, we simply write μ_N^α. We denote by $\mu^{\alpha,\lambda}(\cdot)$ and when $\lambda=0$, $\mu^\alpha(\cdot)$ the measure (2.2) on the infinite product space $\mathcal{S}$. Moreover, for a probability measure μ and a bounded function f, both defined on $\mathcal{S}$ or $\mathcal{S}_N$, we denote by $\mathbf{E}^\mu(f)$ the expectation of f with respect to μ. We need to introduce also the canonical measures $\nu_\rho^{\alpha,N}$,

$$\nu_\rho^{\alpha,N}(\cdot) = \mu_N^{\alpha,\lambda}\left(\cdot \,\middle|\, \sum_{x\in\Lambda_N} \eta_x = \rho|\Lambda_N|\right)$$

for $\rho \in [0, \frac{1}{|\Lambda_N|}, \dots, 1]$. It is well known that the canonical and the grand canonical measures are closely related if the chemical potential λ is chosen canonical conjugate to the density ρ, in the sense that the average density with respect to $\mu_N^{\alpha,\lambda}$ is equal to ρ. As in [8] one can define the random empirical chemical potential and the *annealed chemical potential* $\lambda_0(\rho)$. To our aim it is enough to consider $\lambda_0(\rho)$. For $\rho \in [0, 1]$, the function $\lambda_0(\rho)$ is defined as the unique λ so that

$$\mathbb{E}\left[\int \eta(0) d\mu^{\alpha,\lambda}(\eta)\right] = \mathbb{E}\left[\frac{e^{\alpha(0)+\lambda}}{1+e^{\alpha(0)+\lambda}}\right] = \rho. \tag{2.3}$$

We will consider as reference measure the random Bernoulli product measure $\nu_{\rho(\cdot)}^{\alpha,N}$ on $\mathcal{S}_N$ defined for positive profile $\rho : \Lambda \to (0, 1)$ by

$$\nu_{\rho(\cdot)}^{\alpha,N}(\eta) = \prod_{x\in\Lambda_N} \left\{\frac{e^{[\alpha(x)+\lambda_0(\rho(x/N))]\eta(x)}}{e^{[\alpha(x)+\lambda_0(\rho(x/N))]}+1}\right\}, \tag{2.4}$$

if $\rho(\cdot) \equiv \rho$ is constant, we shall denote simply $\nu_{\rho(\cdot)}^{\alpha,N} = \nu_\rho^{\alpha,N}$. We denote by $\eta^{x,y}$ the configuration obtained from η by interchanging the values at x and y:

$$\eta^{x,y}(z) = \begin{cases} \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x, \\ \eta(z) & \text{otherwise,} \end{cases} \tag{2.5}$$

and by η^x the configuration obtained from η by flipping the occupation number at site x:

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x, \\ 1-\eta(x) & \text{if } z = x. \end{cases} \tag{2.6}$$

Further, for $f : \mathcal{S}_N \to \mathbb{R}$, $x, y \in \Lambda_N$, we denote

$$(\nabla_{x,y} f)(\eta) = f(\eta^{x,y}) - f(\eta).$$

The disordered exclusion process on Λ_N with random reservoirs at its boundary Γ_N is the Markov process on $\mathcal{S}_N$ whose generator $\mathcal{L}_N$ can be decomposed as

$$\mathcal{L}_N = \mathcal{L}_N^0 + \mathcal{L}_N^b, \tag{2.7}$$

where the generators $\mathcal{L}_N^0$, $\mathcal{L}_N^b$ act on function $f : \mathcal{S}_N \to \mathbb{R}$ as

$$(\mathcal{L}_N^0 f)(\eta) = \sum_{e\in\mathcal{E}} \sum_{x\in\Lambda_N, x+e\in\Lambda_N} C(x, x+e; \eta)[(\nabla_{x,x+e} f)(\eta)], \tag{2.8}$$

where e is a generic element of $\mathcal{E}$, the rate

$$C(x,y;\eta) \equiv C^{\alpha}(x,y;\eta) = \exp\left\{-\frac{(\nabla_{x,y}H^{\alpha})(\eta)}{2}\right\}; \tag{2.9}$$

and

$$(\mathcal{L}_N^b f)(\eta) = \sum_{x\in\Gamma_N} C^b(x/N,\eta)\big[f(\eta^x) - f(\eta)\big]. \tag{2.10}$$

To define the rate $C^b(x/N,\eta)$ we fix a function $b(\cdot)$ on Γ, representing the density of the reservoirs. We assume that $b(\cdot)$ is the restriction on Γ of a smooth function $\gamma(\cdot)$ defined on a neighborhood V of Λ, $\gamma : V \to (0,1)$, $\gamma \in C^2(V)$ and $\gamma(u) = b(u)$ for $u \in \Gamma$. The rate C^b is chosen so that $\mathcal{L}_N^b$ is reversible with respect to $\nu_{\gamma(\cdot)}^{\alpha,N}$

$$C^b(x/N,\eta) = \eta(x)\exp\left\{-\frac{\alpha(x)+\lambda_0(b(\frac{x}{N}))}{2}\right\} + (1-\eta(x))\exp\left\{\frac{\alpha(x)+\lambda_0(b(\frac{x}{N}))}{2}\right\}. \tag{2.11}$$

The first term in (2.11) is the creation rate, the second one is the annihilation rate. Next we recall the relevant properties of $C(x,y;\eta)$:

(a) *detailed balance condition with respect to the measure* (2.2);
(b) *positivity and boundedness*: there exists $a > 0$ such that

$$a^{-1} \le C(x,y;\eta) \le a; \tag{2.12}$$

(c) *translation covariant*:

$$C^{\alpha}(x,y;\eta) = C^{\tau_z\alpha}(x-z,y-z;\tau_z\eta) = \tau_z C^{\alpha}(x-z,y-z;\eta), \tag{2.13}$$

where for z in $\mathbb{Z}^d$, τ_z denotes the space shift by z units on $\mathcal{S}\times\Sigma_D$ defined for all $\eta\in\mathcal{S}$, $\alpha\in\Sigma_D$ and $g:\mathcal{S}\times\Sigma_D\to\mathbb{R}$ by

$$(\tau_z\eta)(x) = \eta(x+z), \qquad (\tau_z\alpha)(x) = \alpha(x+z), \qquad (\tau_z g)(\eta,\alpha) = g(\tau_z\eta,\tau_z\alpha). \tag{2.14}$$

We omit to write in the notation the explicit dependence on the randomness α, unless there is an ambiguity. The process arising from the full generator (2.7) is then a superposition of a dynamics with a conservation law (the Kawasaki random dynamics) acting on the whole Λ_N and a birth and death process acting on Γ. Remark that if $b(\cdot)\equiv b_0$ for some positive constant b_0, then the generator $\mathcal{L}_N$, see (2.7), is self-adjoint in $L^2(\nu_{b_0}^{\alpha,N})$ and the measure $\nu_{b_0}^{\alpha,N}$ is the stationary measure for the full dynamics $\mathcal{L}_N$. In the general case, when $b(\cdot)$ is not constant, since the Markov process on $\mathcal{S}_N$ with generator (2.7), is irreducible for all $N\ge 1$, there exists always an unique invariant measure but in general cannot be written in an explicit form.

2.2 The Macroscopic Equation

The macroscopic evolution of the local particles density ρ is described by the quasi-linear parabolic equation

$$\begin{cases} \partial_t\rho = \nabla\cdot(D(\rho)\nabla\rho), \\ \rho(0,\cdot) = \rho_0, \\ \rho(t,\cdot)|_{\Gamma} = b(\cdot) \quad \text{for } t>0, \end{cases} \tag{2.15}$$

where $D(\rho)$ is the diffusion matrix given in (2.17), $b(\cdot) \in C^2(\Gamma)$ and $\rho_0 : \Lambda \to [0,1]$ is the initial profile. The diffusion matrix $D(\cdot)$ is the one derived in [8]. To define it, let [1]

$$\mathbb{G} \equiv \{g : \mathcal{S} \times \Lambda_D \to \mathbb{R};\ \text{local and bounded}\}, \tag{2.16}$$

and for $g \in \mathbb{G}$, $\Gamma_g(\eta) = \sum_{x\in\mathbb{Z}^d} \big(\tau_x g\big)(\eta,\alpha)$. The $\Gamma_g(\eta)$ is a formal expression, but the difference $\nabla_{0,e}\Gamma_g(\eta) = \Gamma_g(\eta^{0,e}) - \Gamma_g(\eta)$ for $e \in \mathcal{E}$ is meaningful. For each $\rho \in (0,1)$, let $D(\rho) = \{D_{i,j}(\rho),\ 1 \le i,j \le d\}$ be the symmetric matrix defined, for every $a \in \mathbb{R}^d$, by the variational formula

$$(a \cdot D(\rho)a) = \frac{1}{2\chi(\rho)} \inf_{g\in\mathbb{G}} \sum_{i=1}^{d} \mathbb{E}\big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\big(C(0,e_i;\eta)\big\{a_i\nabla_{0,e_i}\eta(0) + (\nabla_{0,e_i}\Gamma_g)(\eta)\big\}^2\big)\big], \tag{2.17}$$

where $\lambda_0(\rho)$ is defined in (2.3), $\chi(\rho)$ is the static compressibility given by

$$\chi(\rho) = \mathbb{E}\bigg[\int \eta(0)^2 d\mu^{\alpha,\lambda_0(\rho)}(\eta) - \bigg(\int \eta(0) d\mu^{\alpha,\lambda_0(\rho)}(\eta)\bigg)^2\bigg], \tag{2.18}$$

for $a, b \in \mathbb{R}^d$, $(a \cdot b)$ is the scalar vector product of a and b and $\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}(\cdot)$ is the expectation with respect to $\mu^{\alpha,\lambda_0(\rho)}$, see after (2.2), the random Bernoulli product measure on $\mathcal{S}$ with annealed chemical potential $\lambda_0(\rho)$. In Theorem 2.1 of [8] it has been proved, for $d \ge 3$ and for $\rho \in (0,1)$, the existence of the symmetric diffusion matrix defined in (2.17). Further it has been proved that the coefficients $D_{i,j}(\cdot)$ are nonlinear continuous functions in the open interval $(0,1)$ and there exists a constant $C > 1$, depending on dimensions and bound on the random field, such that

$$\frac{\mathbb{1}}{C} \le D(\rho) \le C\mathbb{1} \quad \rho \in (0,1), \tag{2.19}$$

where $\mathbb{1}$ is the $d \times d$ identity matrix. Quastel [20], proved that the bulk diffusion is continuous on $[0,1]$ and $\frac{1}{2}$-Holder continuous on the open interval. One expects the matrix $D(\cdot)$ to be a smooth function of ρ [12]. Methods has been developed to prove higher regularity for the bulk diffusion, see [1, 18], but their application to this model looks rather difficult.

We will assume all through the paper that $D(\cdot)$ is continuous in $[0,1]$ and Lipschitz in the open interval.

By weak solution of (2.15) we mean a function $\rho(\cdot,\cdot) : [0,T] \times \Lambda \to \mathbb{R}$ satisfying:

(IB1) $\rho \in L^2((0,T); H^1(\Lambda))$:

$$\int_0^T ds \bigg(\int_\Lambda \| \nabla\rho(s,u) \|^2 du\bigg) < \infty. \tag{2.20}$$

(IB2) For every function $G(t,u) = G_t(u)$ in $\mathcal{C}_c^{1,2}([0,T] \times \mathring{\Lambda})$, where $\mathring{\Lambda} =]-1,1[\times \mathbf{T}^{d-1}$ and $\mathcal{C}_c^{1,2}([0,T] \times \mathring{\Lambda})$ is the space of functions from $[0,T] \times \mathring{\Lambda}$ to $\mathbb{R}$ twice continuously

[1] A function $g : \mathcal{S} \times \Lambda_D \to \mathbb{R}$ is local if the support of g, Δ_g, i.e. the smallest subset of $\mathbb{Z}^d$ such that g depends only on $\{(\eta(x), \alpha(x))\ x \in \Delta_g\}$, is finite. The function g is bounded if $\sup_\eta \sup_\alpha |g(\eta,\alpha)| < \infty$.

differentiable in Λ with continuous time derivative and having compact support in $\mathring{\Lambda}$ we have

$$\int_\Lambda du\{G_T(u)\rho(T,u) - G_0(u)\rho(0,u)\} - \int_0^T ds \int_\Lambda du(\partial_s G_s)(u)\rho(s,u)$$
$$= -\int_0^T ds \left\{ \int_\Lambda du\, D(\rho(s,u))\nabla\rho(s,u)\cdot\nabla G_s(u) \right\}.$$

(IB3) For any $t \in (0,T]$, $\mathrm{Tr}(\rho(t,\cdot)) = b(\cdot)$, a.e., where the trace operator $\mathrm{Tr}(\cdot)$ is the linear operator from $H^1(\Lambda)$ to $L^2(\Gamma)$ defined as the continuous extension of the operator which associates to any function $G \in C(\Lambda)$ its boundary value: $\mathrm{Tr}(G) = G|_\Gamma$, see [5].

(IB4) $\rho(0,u) = \rho_0(u)$ a.e.

Notice that, since the original particle model cannot have more than one particle at a lattice site any solution ρ of (2.15) is between 0 and 1. The existence and uniqueness of the weak solution of (2.15) when (2.19) holds and $D(\cdot)$ is Lipschitz continuous for $\rho \in (0,1)$, can be done using standard analysis tools, see [16]. In the appendix a proof of the existence and of uniqueness is provided as consequence of the existence of the hydrodynamic limit and the comparison theorem proved for solutions of (2.15).

Stationary Solution We denote by $\bar\rho$ the stationary solution of (2.15), i.e. a function from $\Lambda \to [0,1]$ so that $\bar\rho \in H^1(\Lambda)$ and for $G \in \mathcal{C}_c^2(\mathring{\Lambda})$ we have

$$\begin{cases} \int_\Lambda du\, D(\bar\rho(u))\nabla\bar\rho(u)\cdot\nabla G(u) = 0, \\ \mathrm{Tr}(\bar\rho(\cdot)) = b(\cdot), \quad \text{a.e.} \end{cases} \tag{2.21}$$

Existence of the weak solution of (2.21) when (2.19) holds, $D(\cdot)$ Lipschitz continuous and $b(\cdot)$ smooth is obtained applying standard analysis tools, see for example [10]. We sketch it in Proposition A.9.

2.3 The Main Results

For any $T > 0$, we denote by $(\eta_t)_{t\in[0,T]}$ the Markov process on $\mathcal{S}_N$ with generator $N^2\mathcal{L}_N$ starting from $\eta_0 = \eta$ and by $\mathbf{P}_\eta := \mathbf{P}_\eta^\alpha$ its distribution when the initial configuration is η. We remind that we omit to write explicitly the dependence on α. The $\mathbf{P}_\eta$ is a probability measure on the path space $D([0,T],\mathcal{S}_N)$, which we consider endowed with the Skorohod topology and the corresponding Borel σ-algebra. Expectation with respect to $\mathbf{P}_\eta$ is denoted by $\mathbf{E}_\eta$. If μ^N is a probability measure on $\mathcal{S}_N$ we denote $\mathbf{P}_{\mu^N}(\cdot) = \int_{\mathcal{S}_N} \mathbf{P}_\eta(\cdot)\mu^N(d\eta)$ and by $\mathbf{E}_{\mu^N}$ the expectation with respect to $\mathbf{P}_{\mu^N}$. For $\eta \in \mathcal{S}_N$, denote by $\pi^N = \pi^N(du;\eta)$ the empirical measure defined by

$$\pi^N = \frac{1}{N^d}\sum_{x\in\Lambda_N} \eta(x)\,\delta_{x/N}(du), \tag{2.22}$$

where $\delta_u(\cdot)$ stands for the Dirac measure on Λ concentrated on u. Since $\eta(x) \in \{0,1\}$, relation (2.22) induces from $\mathbf{P}_{\mu^N}$ a distribution Q_{μ^N} on the Skorohod space $D([0,T],\mathcal{M}_1(\Lambda))$, where $\mathcal{M}_1(\Lambda)$ is the set of positive Borel measures on Λ with total mass bounded by 1, endowed with the weak topology. Denote by $\mathcal{M}_1^0(\Lambda)$ the subset of $\mathcal{M}_1(\Lambda)$ of all absolutely continuous measures w.r.t. the Lebesgue measure with density bounded by 1:

$$\mathcal{M}_1^0(\Lambda) = \{\pi \in \mathcal{M}_1(\Lambda) : \pi(du) = \rho(u)du \text{ and } 0 \le \rho(u) \le 1 \text{ a.e. }\},$$

$\mathcal{M}_1^0(\Lambda)$ is a closed subset of $\mathcal{M}_1(\Lambda)$ endowed with the weak topology and $D([0,T],\mathcal{M}_1^0(\Lambda))$ is a closed subset of $D([0,T],\mathcal{M}_1(\Lambda))$ for the Skorohod topology. The space $\mathcal{M}_1(\Lambda)$ is compact under the topology of weak convergence. For a measure $\pi \in \mathcal{M}_1(\Lambda)$ and a continuous function $G:\Lambda \to \mathbb{R}$ we denote by $\langle \pi, G\rangle$ the integral of G with respect to π

$$\langle \pi, G\rangle = \int_\Lambda du G(u)\pi(du).$$

To state next theorem we need the following definition.

Definition Given $\rho(u)du \in \mathcal{M}_1^0(\Lambda)$, a sequence of probability measures $(\mu^N)_{N\geq 0}$ on $\mathcal{S}_N$ is said to correspond to the macroscopic profile ρ if, for any smooth function G and $\delta > 0$

$$\lim_{N\to\infty} \mu^N \left\{ \left| \frac{1}{N^d} \sum_{x\in\Lambda_N} G(x/N)\eta(x) - \int_\Lambda G(u)\rho(u)du \right| > \delta \right\} = 0. \tag{2.23}$$

Theorem 2.1 *Let $d \geq 3$ and assume that $D(\rho)$ can be continuously extended to the closed interval $[0,1]$. Let μ^N be a sequence of probability measures on $\mathcal{S}_N$ corresponding to the initial profile ρ_0. Then, $\mathbb{P}$-a.s. the sequence of probability measures $(Q_{\mu^N})_{N\geq 0}$ is tight and all its limit points Q^* are concentrated on $\rho(t,u)\mathrm{d}u$, whose densities are weak solutions of (2.15). Moreover if $D(\cdot)$ is Lipschitz continuous for $\rho \in (0,1)$, then $(Q_{\mu^N})_{N\geq 0}$ converges weakly, as $N \uparrow \infty$, to Q^*. This limit point is concentrated on the unique weak solution of (2.15).*

Denote by $\nu_s^{\alpha,N}$ the unique invariant measure of the Markov process $(\eta_t)_{t\in[0,T]}$ with generator $N^2\mathcal{L}_N$. We have the following:

Theorem 2.2 *Let $d \geq 3$, assume that $D(\rho)$ can be continuously extended to the closed interval $[0,1]$ and Lipschitz continuous for $\rho \in (0,1)$. For every continuous function $G:\Lambda \to \mathbb{R}$ and every $\delta > 0$,*

$$\lim_{N\to\infty} \nu_s^{\alpha,N} \left\{ \left| \frac{1}{N^d} \sum_{x\in\Lambda_N} G(x/N)\eta(x) - \int_\Lambda G(u)\bar\rho(u)du \right| > \delta \right\} = 0, \quad \mathbb{P}\text{-}a.e., \tag{2.24}$$

where $\bar\rho(\cdot)$ is the unique solution of (2.21).

3 Strategy of Proof and Basic Estimates

3.1 The Steps to Prove Theorem 2.1

To prove the hydrodynamic behavior of the system we follow the non gradient method developed by [8] for this model, based on the Varadhan paper [23] and the entropy method introduced by [11]. As explained in the introduction, since the reference invariant state is not explicitly known, we compute the entropy of the state of the process with respect to a product measure with slowly varying profile $\gamma(\cdot)$. We prove in Lemma 3.8 that, provided $\gamma(\cdot)$ is smooth enough and takes the prescribed value $b(\cdot)$ at the boundary, the rate to which the entropy increases is of the order of the volume, N^d, and for finite time T this implies only a modification of the constant multiplying N^d.

We divide the proof of the hydrodynamic behavior in three steps: tightness of the measures $(Q_{\mu^N})_{N\geq 1}$, energy estimates and identification of the support of Q^* as weak solution of (2.15) with fixed boundary conditions. We then refer to [13], Chap. IV, that presents arguments, by now standard, to deduce the hydrodynamic behavior of the empirical measures from the preceding results and the uniqueness of the weak solution of (2.15). We state without proving the first two steps, tightness of the measures and energy estimates. The proof of them can be easily derived from results already in the literature, which we refer to, see [8, 17].

Proposition 3.1 (Tightness) *For almost any disorder configuration $\alpha \in \Sigma_D$, the sequence $(Q_{\mu^N})_{N\geq 1}$ is tight and all its limit points Q^* are concentrated on absolutely continuous paths $\pi(t,du) = \rho(t,u)du$ whose density ρ is positive and bounded above by* 1:

$$Q^*\{\pi : \pi(t,du) = \rho(t,u)du\} = 1, \qquad Q^*\{\pi : 0 \leq \rho(t,u) \leq 1\} = 1. \tag{3.1}$$

Tightness for non gradient systems in contact with reservoirs is proven in a way similar to the one for non gradient systems with periodic boundary conditions, see [13], Chap. 7, Sect. 6. The main difference relies on the fact that for systems in contact with reservoirs the invariant states are not product probability measures and some additional argument is required. This can be proven as in [17], Sect. 6.

In the next step we prove that $\mathbb{P}$-a.s. every limit point Q^* of the sequence $(Q_{\mu_N})_{N\geq 1}$ is concentrated on paths whose densities ρ satisfy (2.20).

Proposition 3.2 $\mathbb{P}$-*a.s., every limit points Q^* of the sequence $(Q_{\mu^N})_{N\geq 1}$ is concentrated on the trajectories that satisfies* $(IB1)$.

The proof can be done applying arguments as in Proposition A.1.1. of [14]. However the latter proof requires an application of Feynman-Kac formula, for which we have to replace our dynamic (2.7) (cf. [8]).

We then show that $\mathbb{P}$-a.s. any limit point Q^* is supported on densities ρ satisfying (2.15) in the weak sense. This is proven in Proposition 3.3 and in Proposition 3.4 stated below. Proposition 3.3 takes in account only the bulk dynamics and it is based on the [8] results. The main step to prove it consists in replacing the empirical current defined in (3.17) by a function of the density gradient. The proof of this important point, following [8], is given in Theorem 3.10. Proposition 3.4 takes in account the boundary dynamics. For $\ell \in \mathbb{N}$, $x \in \Lambda_N$, with $-N+\ell \leq x_1 \leq N-\ell$ denote by $\eta^\ell(x)$ the average density of η in a cube of width $2\ell+1$ centered at x

$$\eta^\ell(x) = \frac{1}{(2\ell+1)^d} \sum_{y:|y-x|\leq \ell} \eta(y). \tag{3.2}$$

For a function G on Λ, $e \in \mathcal{E}$, $\partial_e^N G$ denotes the discrete (space) derivative in the direction e

$$(\partial_e^N G)(x/N) = N[G((x+e)/N) - G(x/N)] \quad \text{with } x \text{ and } x+e \in \Lambda_N, \tag{3.3}$$

and to short notation we denote by $\partial_k^N G := \partial_{e_k}^N G$ for $1 \leq k \leq d$.

Proposition 3.3 *Assume that $D(\rho)$ defined in* (2.17) *can be continuously extended in* $[0,1]$. *Then,* $\mathbb{P}$-*a.s., for any function G in $\mathcal{C}_c^{1,2}([0,T]\times \mathring{\Lambda})$ and any $\delta > 0$, we have*

$$\limsup_{c\to 0} \limsup_{a\to 0} \limsup_{N\to\infty} \mathbf{P}_{\mu^N}(|\mathcal{B}_{a,c}^{G,N}| \geq \delta) = 0, \tag{3.4}$$

where

$$
\begin{aligned}
\mathcal{B}_{a,c}^{G,N} &= N^{-d} \sum_{x\in\Lambda_N} G(T, x/N)\eta_T(x) - N^{-d} \sum_{x\in\Lambda_N} G(0, x/N)\eta_0(x) \\
&\quad - N^{-d} \sum_{x\in\Lambda_N} \int_0^T \partial_s G(s, x/N)\eta_s(x)ds \\
&\quad + \sum_{1\le k,m\le d} \int_0^T ds N^{1-d} \sum_{x\in\Lambda_N} (\partial_k^N G)(s, x/N) \\
&\quad \times \Big\{ D_{k,m}(\eta_s^{[aN]}(x))\{(2c)^{-1}[\eta_s^{[aN]}(x+cNe_m) - \eta_s^{[aN]}(x-cNe_m)]\}\Big\}. \qquad (3.5)
\end{aligned}
$$

The proof is given in Sect. 3.3. Note that in the statement of Proposition 3.3 the function G has compact support, so the boundary terms do not enter.

The last step states that $\mathbb{P}$-a.s., any limit points Q^* of the sequence $(Q_{\mu^N})_{N\ge 1}$ is concentrated on the trajectories with fixed density at the boundary and equal to $b(\cdot)$:

Proposition 3.4 *$\mathbb{P}$-a.s., any limit point Q^* of the sequence $(Q_{\mu^N})_{N\ge 1}$ is concentrated on the trajectories that satisfy* (IB3).

The proof is given in Sect. 3.4.

3.2 Basic Estimates

Lemma 3.5 (Ergodic lemma) *Let $V : \Sigma_D \times \Lambda \to \mathbb{R}$ be a bounded function, local with respect to the first variable and continuous with respect to the second variable, that is for any $\alpha \in \Sigma_D$ the function $u \to V(\alpha, u)$ is continuous and there exists an integer $\ell \ge 1$ such that for all $u \in \Lambda$ the support of $V(\cdot, u) \subset \{-\ell, \dots, \ell\}^d$. Then*

$$
\lim_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N} \tau_x V(\alpha, x/N) = \int_\Lambda \mathbb{E}[V(\cdot, u)]du \quad \mathbb{P}\text{-a.s.} \qquad (3.6)
$$

Proof We decompose the left-hand side of the limit (3.6) in two parts

$$
\begin{aligned}
N^{-d} \sum_{x\in\Lambda_N} \tau_x V(\alpha, x/N) &= N^{-d} \sum_{x\in\Lambda_N} (\tau_x V(\alpha, x/N) - \mathbb{E}[V(\cdot, x/N)]) \\
&\quad + N^{-d} \sum_{x\in\Lambda_N} \mathbb{E}[V(\cdot, x/N)] - \int_\Lambda \mathbb{E}[V(\cdot, u)]du.
\end{aligned}
$$

By the stationary of $\mathbb{P}$ and the continuity of $u \to \mathbb{E}\big[V(\cdot, u)\big]$, the second term of the right-hand side of the last equality converges to 0 as $N \to \infty$. The first term converges to 0, from Chebychef inequality and the classical method of moments usually used in the proof of strong law of large numbers. □

We start recalling the definition of relative entropy, which is the main tool in the [11] approach. Let $\nu_{\rho(\cdot)}^{\alpha,N}$ be the product measure defined in (2.4) and μ a probability measure on

$\mathcal{S}_N$. Denote by $H(\mu|\nu^{\alpha,N}_{\rho(\cdot)})$ the relative entropy of μ with respect to $\nu^{\alpha,N}_{\rho(\cdot)}$:

$$H(\mu|\nu^{\alpha,N}_{\rho(\cdot)}) = \sup_f \left\{ \int f(\eta)\mu(d\eta) - \log \int e^{f(\eta)} \nu^{\alpha,N}_{\rho(\cdot)}(d\eta) \right\},$$

where the supremum is carried over all bounded functions on $\mathcal{S}_N$. Since $\nu^{\alpha,N}_{\rho(\cdot)}$ gives a positive probability to each configuration, μ is absolutely continuous with respect to $\nu^{\alpha,N}_{\rho(\cdot)}$ and we have an explicit formula for the entropy:

$$H(\mu|\nu^{\alpha,N}_{\rho(\cdot)}) = \int \log\left\{ \frac{d\mu}{d\nu^{\alpha,N}_{\rho(\cdot)}} \right\} d\mu. \tag{3.7}$$

Further, since there is at most one particle per site, there exists a constant C, that depends only on $\rho(\cdot)$, such that for all $\alpha \in \Sigma_D$

$$H(\mu|\nu^{\alpha,N}_{\rho(\cdot)}) \le C N^d \tag{3.8}$$

for all probability measures μ on $\mathcal{S}_N$ (cf. comments following Remark V.5.6 in [13]). To estimate the entropy of the states of the process with respect to the reference measure we define the following functionals from $L^2(\nu)$ to $\mathbb{R}^+$:

$$\begin{aligned} \mathcal{D}^0_N(h,\nu) &= \frac{1}{2}\sum_{e\in\mathcal{E}} \sum_{x,x+e\in\Lambda_N} \int C(x,x+e;\eta)(h(\eta^{x,x+e}) - h(\eta))^2 d\nu(\eta), \\ \mathcal{D}^b_N(h,\nu) &= \frac{1}{2}\sum_{x\in\Gamma_N} \int C^b(x/N,\eta)(h(\eta^x) - h(\eta))^2 d\nu(\eta). \end{aligned} \tag{3.9}$$

Lemma 3.6 *Let $\gamma : \Lambda \to (0,1)$ be a smooth function such that $\gamma|_\Gamma = b(\cdot)$. For any $\alpha \in \Sigma_D$ and $a > 0$ there exists a positive constant $C_0 \equiv C_0(A, \|\nabla\gamma\|_\infty)$ so that for any $f \in L^2(\nu^{\alpha,N}_{\gamma(\cdot)})$,*

$$\int_{\mathcal{S}_N} f(\eta)\mathcal{L}^0_N f(\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) \le -\left(1 - \frac{1}{2a}\right)\mathcal{D}^0_N(f,\nu^{\alpha,N}_{\gamma(\cdot)}) + C_0 N^{d-2}(a+1)\|f\|^2_{L^2(\nu^{\alpha,N}_{\gamma(\cdot)})}, \tag{3.10}$$

$$\int_{\mathcal{S}_N} f(\eta)\mathcal{L}^b_N f(\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) = -\mathcal{D}^b_N(f,\nu^{\alpha,N}_{\gamma(\cdot)}). \tag{3.11}$$

Proof By (3.9),

$$\begin{aligned} &\int_{\mathcal{S}_N} f(\eta)\mathcal{L}^0_N f(\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) \\ &\quad = -\mathcal{D}^0_N(f,\nu^{\alpha,N}_{\gamma(\cdot)}) \\ &\qquad + \frac{1}{2}\sum_{e\in\mathcal{E}} \sum_{x,x+e\in\Lambda_N} \int C(x,x+e;\eta)(\nabla_{x,x+e} f)(\eta) f(\eta^{x,x+e}) R_1(x,x+e;\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta), \end{aligned}$$

where

$$R_1(x,x+e;\eta) = (\nabla_{x,x+e}\eta(x))(e^{(N^{-1}\partial^N_e \lambda_0(\gamma(x/N)))} - 1).$$

By the elementary inequality $2uv \le au^2 + a^{-1}v^2$ which holds for any $a > 0$, for any $x, x + e \in \Lambda_N$

$$\begin{aligned}
&\int C(x, x+e; \eta)(\nabla_{x,x+e} f) f(\eta^{x,x+e}) R_1(x, x+e, \eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) \\
&\quad \le \frac{1}{2a} \int C(x, x+e; \eta)(\nabla_{x,x+e} f)^2 d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) \\
&\qquad + \frac{a}{2} \int C(x, x+e; \eta) f(\eta^{x,x+e})^2 (R_1(x, x+e))^2 d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta).
\end{aligned}$$

To conclude the proof it remains to use Taylor expansion and an integration by part in the second term of the right-hand side of the last inequality. On the other hand, since $\gamma|_\Gamma = b(\cdot)$ the measure $\nu^{\alpha,N}_{\gamma(\cdot)}$ is reversible with respect to $\mathcal{L}^b_N$. A simple computation shows that

$$\int_{\mathcal{S}_N} f(\eta) \mathcal{L}^b_N f(\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta) = -\mathcal{D}^b_N(f, \nu^{\alpha,N}_{\gamma(\cdot)}).$$

□

Lemma 3.7 *Let $\rho, \rho_0 : \Lambda \to (0,1)$ be two smooth functions. There exists a positive constant $C'_0 \equiv C'_0(A, \|\nabla\rho_0\|_\infty, \|\nabla\rho\|_\infty)$ such that for any probability measure μ^N on $\mathcal{S}_N$ and for any $\alpha \in \Sigma_D$,*

$$\mathcal{D}^0_N\left(\sqrt{\frac{d\mu^N}{d\nu^{\alpha,N}_{\rho(\cdot)}}}, \nu^{\alpha,N}_{\rho(\cdot)}\right) \le 2\mathcal{D}^0_N\left(\sqrt{\frac{d\mu^N}{d\nu^{\alpha,N}_{\rho_0(\cdot)}}}, \nu^{\alpha,N}_{\rho_0(\cdot)}\right) + C'_0 N^{d-2}. \tag{3.12}$$

Proof Denote by $f(\eta) = \frac{d\mu^N}{d\nu^{\alpha,N}_{\rho(\cdot)}}(\eta)$ and $h(\eta) = \frac{d\mu^N}{d\nu^{\alpha,N}_{\rho_0(\cdot)}}(\eta)$. Since $f(\eta) = h(\eta)\frac{d\nu^{\alpha,N}_{\rho_0(\cdot)}(\eta)}{d\nu^{\alpha,N}_{\rho(\cdot)}(\eta)}$ we obtain for $e \in \mathcal{E}$ and $x, x+e \in \Lambda_N$ the following

$$\begin{aligned}
&\int_{\mathcal{S}_N} C(x, x+e; \eta)[\nabla_{x,x+e}\sqrt{f}(\eta)]^2 d\nu^{\alpha,N}_{\rho(\cdot)}(\eta) \\
&\quad = \int_{\mathcal{S}_N} C(x, x+e; \eta)[\sqrt{h}(\eta^{x,x+e}) R_2(x, x+e; \eta) + \nabla_{x,x+e}\sqrt{h}(\eta)]^2 d\nu^{\alpha,N}_{\rho_0(\cdot)}(\eta) \\
&\quad \le 2\int_{\mathcal{S}_N} C(x, x+e; \eta)[\nabla_{x,x+e}\sqrt{h}(\eta)]^2 d\nu^{\alpha,N}_{\rho_0(\cdot)}(\eta) \\
&\qquad + 2\int_{\mathcal{S}_N} C(x, x+e; \eta) h(\eta^{x,x+e})[R_2(x, x+e; \eta)]^2 d\nu^{\alpha,N}_{\rho_0(\cdot)}(\eta),
\end{aligned}$$

where

$$R_2(x, x+e; \eta) = \exp\{(1/2)N^{-1}\partial^N_e[\lambda_0(\rho(x/N)) - \lambda_0(\rho_0(x/N))]\nabla_{x,x+e}\eta(x)\} - 1.$$

We conclude the proof using Taylor expansion and integration by parts. □

Denote by S^N_t the semigroup associated to the generator $N^2\mathcal{L}_N$. Given a probability measures μ^N on $\mathcal{S}_N$ denote by $\mu^N(t)$ the state of the process at time $t : \mu^N(t) = \mu^N S^N_t$. Recall that $\gamma : \Lambda \to (0,1)$ is a smooth profile equal to b at the boundary of Λ. Let h^N_t be the density of $\mu^N(t)$ with respect to $\nu^{\alpha,N}_{\gamma(\cdot)}$. Let $\mathcal{L}^*_{\gamma,N}$ be the adjoint of $\mathcal{L}_N$ in $L^2(\nu^{\alpha,N}_{\gamma(\cdot)})$. It is easy to check that

$$\partial_t h^N_t = N^2 \mathcal{L}^*_{\gamma,N} h^N_t. \tag{3.13}$$

Notice that $\mathcal{L}^*_{\gamma,N}$ is not a generator because $\nu^{\alpha,N}_{\gamma(\cdot)}$ is not an invariant measure for the Markov process with generator $\mathcal{L}_N$. We denote by $H_N(t)$ the entropy of $\mu^N(t)$ with respect to $\nu^{\alpha,N}_{\gamma(\cdot)}$, see (3.7),

$$H_N(t) := H(\mu^N(t)|\nu^{\alpha,N}_{\gamma(\cdot)}). \tag{3.14}$$

Lemma 3.8 *There exists positive constant $C = C(\|\nabla\gamma\|_\infty)$ such that for any $a > 0$ and for any $\alpha \in \Sigma_D$*

$$\partial_t H_N(t) \le -2(1-a)N^2 \mathcal{D}^0_N\left(\sqrt{h^N_t}, \nu^{\alpha,N}_{\gamma(\cdot)}\right) - 2N^2 \mathcal{D}^b_N\left(\sqrt{h^N_t}, \nu^{\alpha,N}_{\gamma(\cdot)}\right) + \frac{C}{a} N^d.$$

Proof By (3.13) and the explicit formula for the entropy we have that

$$\partial_t H_N(t) = N^2 \int_{\mathcal{S}_N} h^N_t \mathcal{L}_N \log(h^N_t) d\nu^{\alpha,N}_{\gamma(\cdot)}.$$

Using the basic inequality $a(\log b - \log a) \le -(\sqrt{a} - \sqrt{b})^2 + (b-a)$ for positive a and b, we obtain

$$\begin{aligned}\partial_t H_N(t) \le{}& -2N^2 \mathcal{D}^0_N\left(\sqrt{h^N_t}, \nu^{\alpha,N}_{\gamma(\cdot)}\right) - 2N^2 \mathcal{D}^b_N\left(\sqrt{h^N_t}, \nu^{\alpha,N}_{\gamma(\cdot)}\right)\\ &+ N^2 \int_{\mathcal{S}_N} \mathcal{L}^0_N h^N_t d\nu^{\alpha,N}_{\gamma(\cdot)} + N^2 \int_{\mathcal{S}_N} \mathcal{L}^b_N h^N_t d\nu^{\alpha,N}_{\gamma(\cdot)}.\end{aligned} \tag{3.15}$$

Since $\gamma(u) = b(u)$ for $u \in \Gamma$, $\nu^{\alpha,N}_{\gamma(\cdot)}$ is reversible with respect to $\mathcal{L}^b_N$. This implies that

$$\int_{\mathcal{S}_N} \mathcal{L}^b_N h^N_t d\nu^{\alpha,N}_{\gamma(\cdot)} = 0.$$

Next we bound $\int_{\mathcal{S}_N} \mathcal{L}^0_N h^N_t d\nu^{\alpha,N}_{\gamma(\cdot)}$ in terms of $\mathcal{D}^0_N$. Denote by $R : \mathbb{R} \to \mathbb{R}$ the function defined by $R(u) = e^u - 1 - u$. A standard computation shows that

$$\begin{aligned}&N^2 \int_{\mathcal{S}_N} \mathcal{L}^0_N h^N_t d\nu^{\alpha,N}_{\gamma(\cdot)}\\ &\quad= N^2 \sum_{e\in\mathcal{E}} \sum_{x,x+e\in\Lambda_N} \int C(x,x+e;\eta) h^N_t(\eta) R(N^{-1}\partial^N_e \lambda_0(\gamma(x/N))\nabla_{x,x+e}\eta(x)) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta)\\ &\qquad+ N \sum_{e\in\mathcal{E}} \sum_{x,x+e\in\Lambda_N} (\partial^N_e \lambda_0(\gamma(x/N))) \int W_{x,x+e}(\eta) h^N_t(\eta) d\nu^{\alpha,N}_{\gamma(\cdot)}(\eta),\end{aligned} \tag{3.16}$$

where $W_{x,x+e}(\eta)$ is the current over the bond $(x, x+e)$:

$$W_{x,x+e}(\eta) \equiv C(x,x+e;\eta)[\eta(x) - \eta(x+e)]. \tag{3.17}$$

We will often omit to write the dependence of $W_{x,x+e}(\eta)$ on η. By Taylor expansion and the elementary inequality $|R(u)| \le \frac{u^2}{2} e^{|u|}$, we obtain using the fact that γ is smooth and h_t^N is a probability density with respect to $\nu^{\alpha,N}_{\gamma(\cdot)}$, that the first term of the right-hand side of (3.16) is bounded by $C\,N^d$ for some positive constant C. On the other hand integrating by part, applying the same computations as in Lemma 5.1 of [17], we obtain that there exists a constant $C_0 = C(\|\nabla\gamma\|_\infty)$ so that for any $a > 0$

$$\int W_{x,x+e} h_t^N d\nu^{\alpha,N}_{\gamma(\cdot)} \le \frac{1}{a} \int C(x, x+e; \eta)\Big(\nabla_{x,x+e}\sqrt{h_t^N}\Big)^2 d\nu^{\alpha,N}_{\gamma(\cdot)} + C_0\{a + N^{-1}\}$$

for $x, x+e \in \Lambda_N$. □

For $z \in \Lambda_N$, $M \in \mathbb{N}$ denote by $\Lambda_M(z)$ the intersection of a cube centered at $z \in \Lambda_N$ of edge $2M+1$ with Λ_N, i.e.

$$\Lambda_M(z) := \{z + \Lambda_M\} \cap \Lambda_N. \tag{3.18}$$

For probability measure ν^N on $\mathcal{S}_N$, denote by $\mathcal{D}^0_{M,z}(\cdot\,, \nu^N)$ the Dirichlet form corresponding to jumps in $\Lambda_M(z)$:

$$\mathcal{D}^0_{M,z}(f, \nu^N) = \frac{1}{2} \sum_{x,x+e \in \Lambda_M(z)} \int C(x, x+e; \eta)(\nabla_{x,x+e} f(\eta))^2 d\nu^N(\eta). \tag{3.19}$$

Similarly, for $z \in \Gamma_N$ define $\mathcal{D}^b_{M,z}(\cdot\,, \nu^N)$ the Dirichlet form corresponding to creation and destruction of particles at sites in Γ_N which are at distance less than M from z:

$$\mathcal{D}^b_{M,z}(f, \nu^N) = \frac{1}{2} \sum_{x \in \Gamma_N \cap \Lambda_M(z)} \int C^b(x/N, \eta)(f(\eta^x) - f(\eta))^2 d\nu^N(\eta). \tag{3.20}$$

Fix any $z \in \Gamma_N$ denote by $f_t^{z,N}$ the Radon-Nikodym derivative of $\mu^N(t)$ with respect to $\nu^{\alpha,N}_{b(z/N)}$, the random Bernoulli measure on $\mathcal{S}_N$ with constant parameter equal to $b(\frac{z}{N})$. Recall that we denoted by h_t^N the Radon-Nikodym derivative of $\mu^N(t)$ with respect to $\nu^{\alpha,N}_{\gamma(\cdot)}$ and that $b(\frac{z}{N}) = \gamma(\frac{z}{N})$ for $z \in \Gamma$. We have the following result.

Lemma 3.9 *Take $M \in \mathbb{N}$, $M < N$. There exists a positive constant $C_0 = C(\|\nabla\gamma\|_\infty)$ depending only on $\gamma(\cdot)$ such that for any $z \in \Gamma_N$*

$$\mathcal{D}^0_{M,z}\Big(\sqrt{f_t^{z,N}}, \nu^{\alpha,N}_{b(z/N)}\Big) \le 2\mathcal{D}^0_{M,z}\Big(\sqrt{h_t^N}, \nu^{\alpha,N}_{\gamma(\cdot)}\Big) + C_0 \frac{M^d}{N^2},$$
$$\mathcal{D}^b_{M,z}\Big(\sqrt{f_t^{z,N}}, \nu^{\alpha,N}_{b(z/N)}\Big) \le 2\mathcal{D}^b_{M,z}\Big(\sqrt{h_t^N}, \nu^{\alpha,N}_{\gamma(\cdot)}\Big) + C_0 \frac{M^{d+1}}{N^2}.$$

The proof is similar to the proof of Lemma 3.7.

3.3 Proof of Proposition 3.3

We prove in this section Proposition 3.3. Let Q^* be a limit point of the sequence $(Q_{\mu^N})_{N\ge 1}$ and assume, without loss of generality, that $\mathbb{P}$-a.s., Q_{μ^N} converges to Q^*. Fix a function G in $\mathcal{C}^{1,2}_c([0,T]\times \mathring{\Lambda})$. For $\alpha \in \Sigma_D$ consider the $\mathbf{P}_{\mu^N}$ martingales with respect to the natural filtration associated with $(\eta_t)_{t\in[0,T]}$, $M_t^G \equiv M_t^{G,N,\alpha}$ and $\mathcal{N}_t^G \equiv \mathcal{N}_t^{G,N,\alpha}$, $t \in [0,T]$, defined by

$$\begin{aligned} M_t^G &= \langle \pi_t^N, G_t\rangle - \langle \pi_0^N, G_0\rangle - \int_0^t (\langle \pi_s^N, \partial_s G_s\rangle + N^2 \mathcal{L}_N \langle \pi_s^N, G_s\rangle)\,ds, \\ \mathcal{N}_t^G &= (M_t^G)^2 - \int_0^t \{N^2 \mathcal{L}_N(\langle \pi_s^N, G_s\rangle)^2 - 2\langle \pi_s^N, G_s\rangle N^2 \mathcal{L}_N \langle \pi_s^N, G_s\rangle\} ds. \end{aligned} \tag{3.21}$$

A computation of the integral term of $\mathcal{N}_t^G$ shows that the expectation of the quadratic variation of M_t^G vanishes as $N \uparrow 0$. Therefore, by Doob's inequality, for every $\delta > 0$, $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty} \mathbf{P}_{\mu_N}[\sup_{0\le t\le T} |M_t^G| > \delta] = 0. \tag{3.22}$$

By (2.13) and since for any $s \in [0,T]$ the function G_s has compact support in $\mathring{\Lambda}$, a summation by parts permits to rewrite the integral term of M_t^G as

$$\int_0^t \langle \pi_s^N, \partial_s G_s\rangle ds + \int_0^t \left\{ N^{1-d} \sum_{k=1}^d \sum_{x\in\Lambda_N} (\partial_k^N G_s)(x/N) W_{x,x+e_k}(\eta_s) \right\} ds, \tag{3.23}$$

where the current $W_{x,x+e_k}$ is defined in (3.17). To localize the dynamics define for any $0 < r < 1$

$$\begin{aligned} &\Lambda_r = [-r,r] \times \mathbf{T}^{d-1}, \qquad \Lambda_{rN} = \{(x_1,\dots,x_d) \in \Lambda_N : -rN \le x_1 \le rN\}, \\ &\Gamma_{rN} = \{x \in \Lambda_{rN} : x_1 = \pm rN\}. \end{aligned} \tag{3.24}$$

Set, for $0 < a < c < 1$, $k = 1,\dots,d$,

$$\mathbb{V}_k^{N,c,a}(\eta,\alpha) = N W_{0,e_k} + \sum_{m=1}^d D_{k,m}\left(\eta^{[aN]}(0)\right)\left\{(2c)^{-1}\left[\eta^{[aN]}(cNe_m) - \eta^{[aN]}(-cNe_m)\right]\right\}. \tag{3.25}$$

Next theorem is the main step in the proof of Proposition 3.3.

Theorem 3.10 *Assume that $D(\cdot)$ defined in (2.17) can be continuously extended in* $[0,1]$. *Then*, $\mathbb{P}$*-a.s., for any* $G \in \mathcal{C}_c^{1,2}([0,T]\times \mathring{\Lambda})$,

$$\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{N\to\infty} \mathbf{E}_{\mu_N}\left[\left| N^{-d} \int_0^T \sum_{x\in\Lambda_N} G_s(x/N)\tau_x \mathbb{V}_k^{N,c,a}(\eta_s,\alpha)\,ds \right|\right] = 0 \tag{3.26}$$

for $k = 1,\dots,d$.

Proof As in the proof of Theorem 3.2, Sect. 4, of [8], by the regularity of the test function G, we first replace the current $W_{x,x+e_k}$ appearing in $\tau_x \mathbb{V}_k^{N,c,a}(\eta_s,\alpha)$ by its local mean around x. More precisely for any $\ell \ge 1$ we have

$$\lim_{N\to\infty} \mathbf{E}_{\mu_N}\left[\left| N^{-d+1} \int_0^T \sum_{x\in\Lambda_N} G_s(x/N)\left[W_{x,x+e_k} - \tilde{W}^{\ell}_{x,x+e_k}\right] ds \right|\right] = 0,$$

where $\tilde{W}^{\ell}_{x,x+e_k}$ is the local mean of the current

$$\tilde{W}^{\ell}_{x,x+e_k} = \frac{1}{(2\ell_1+1)^d} \sum_{|y-x|\le \ell_1} W_{y,y+e_k}$$

and $\ell_1 = \ell - \sqrt{\ell}$.

The second step is to note that, since $G_s(\cdot)$ has compact support in $\mathring{\Lambda}$ for all $s \in [0,T]$, we have

$$\sum_{x\in\Lambda_N} G_s(x/N)\tau_x\left(\mathcal{L}^b_N g\right) = 0,$$

for any local function $g \in \mathbb{G}$, see (2.16). Then, by martingale methods $\mathbb{P}$-a.s.,

$$\limsup_{N\to\infty} \mathbf{E}_{\mu_N}\left[\left|\int_0^T ds\left(N^{-d+1}\sum_{x\in\Lambda_N} G_s(x/N)\tau_x\mathcal{L}^0_N g\right)\right|\right] = 0. \tag{3.27}$$

By the regularity of G_s as done before, we can replace $\tau_x\mathcal{L}^0_N g$ by its local mean.

Let $0<\theta<1$ such that for any $t\in[0,T]$, the support of the function G_t is a subset of $\Lambda_{(1-2\theta)}$. Fix a smooth function $\gamma_\theta\colon \Lambda\to(0,1)$ which coincides with b at the boundary of Λ and constant inside $\Lambda_{(1-\theta)}$. Denote by

$$\begin{aligned}\tilde{\mathbb{V}}^{N,c,a}_{k,\ell,g}(\eta_s) &= N\tilde{W}^{\ell}_{0,e_k} + N\frac{1}{(2\ell_1+1)^d}\sum_{|y|\le\ell_1}\tau_y(\mathcal{L}^0_N g)\\ &\quad + \sum_{m=1}^d D_{k,m}\left(\eta^{[aN]}(0)\right)\left\{(2c)^{-1}\left[\eta^{[aN]}(cNe_m) - \eta^{[aN]}(-cNe_m)\right]\right\},\end{aligned}$$

and by $Z^{N,c,a}_{k,\ell,g}(G,\eta)$ the quantity

$$Z^{N,c,a}_{k,\ell,g}\big(G,\eta\big) = N^{-d}\sum_{x\in\Lambda_N} G(x/N)\tau_x\tilde{\mathbb{V}}^{N,c,a}_{k,\ell,g}(\eta).$$

The proof of (3.26) is achieved once we show that

$$\inf_{g\in\mathbb{G}}\ \limsup_{c\downarrow 0,a\downarrow 0,\ell\uparrow\infty,N\uparrow\infty} \mathbf{E}_{\mu_N}\left[\left|\int_0^T Z^{N,c,a}_{k,\ell,g}\big(G_s,\eta_s,\big)\,ds\right|\right] = 0$$

for $k=1,\dots,d$. Since the entropy of μ^N with respect to $\nu^{\alpha,N}_{\gamma_\theta(\cdot)}$ is bounded by $C_\theta|\Lambda_N|$ for some finite constant C_θ, by the entropy inequality

$$\begin{aligned}&\mathbf{E}_{\mu_N}\left[\left|\int_0^T Z^{N,c,a}_{k,\ell,g}\big(G_s,\eta_s\big)\,ds\right|\right]\\ &\quad\le \frac{C_\theta}{B} + \frac{1}{BN^d}\log\mathbf{E}_{\nu^{\alpha,N}_{\gamma_\theta(\cdot)}}\left[\exp\left\{BN^d\left|\int_0^T Z^{N,a,c}_{k,\ell,g}\big(G_s,\eta_s\big)ds\right|\right\}\right]\end{aligned} \tag{3.28}$$

for any positive B. Since $e^{|x|}\le e^x + e^{-x}$ and

$$\limsup N^{-d}\log\{a_N+b_N\} \le \max\{\limsup N^{-d}\log a_N,\ \limsup N^{-d}\log b_N\},$$

we may remove the absolute value in the second term of (3.28), provided our estimate remains in force if we replace G by $-G$. By the Feynman-Kac formula,

$$\frac{1}{BN^d}\log \mathbf{E}_{\nu^{\alpha,N}_{\gamma_\theta(\cdot)}}\left[\exp\left\{BN^d\int_0^T Z^{N,a,c}_{k,\ell,g}(G_s,\eta_s)ds\right\}\right]\le \frac{1}{BN^d}\int_0^T \lambda^{\ell,g}_{N,c,a}(G_s)\,ds,$$

where $\lambda^{\ell,g}_{N,c,a}(G_s)$ is the largest eigenvalue of the $N^2\{\mathcal{L}^{sym}_N + BZ^{N,c,a}_{k,\ell,g}(G_s,\eta)\}$ where $\mathcal{L}^{sym}_N := \frac{1}{2}(\mathcal{L}_N+\mathcal{L}^*_{\gamma_\theta,N})$ and $\mathcal{L}^*_{\gamma_\theta,N}$ is the adjoint of $\mathcal{L}_N$ in $L^2(\nu^{\alpha,N}_{\gamma_\theta(\cdot)})$. By the variational formula for the largest eigenvalue, for $s\in[0,T]$, we have that

$$\frac{1}{BN^d}\lambda^{\ell,g}_{N,c,a}(G_s)=\sup_f\left\{\int Z^{N,c,a}_{k,\ell,g}(G_s,\eta)f(\eta)\nu^{\alpha,N}_{\gamma_\theta(\cdot)}(d\eta)+\frac{N^{2-d}}{B}\langle \mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\gamma_\theta(\cdot)}\right\}.$$

In this formula the supremum is carried over all densities f with respect to $\nu^{\alpha,N}_{\gamma_\theta(\cdot)}$. Since $\gamma_\theta(\cdot)$ coincides with $b(\cdot)$ on Γ, $\mathcal{L}^b_N$ is reversible with respect to $\gamma_\theta(\cdot)$, so that $\langle\mathcal{L}^b_N\sqrt{f},\sqrt{f}\rangle_{\gamma_\theta(\cdot)}$ is negative. We then apply (3.10) of Lemma 3.6 to estimate $\langle \mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\gamma_\theta(\cdot)}$ by $-(1/2)\mathcal{D}^0_N(\sqrt{f},\nu^{\alpha,N}_{\gamma_\theta(\cdot)})+C'_\theta N^{d-2}$ for some constant C'_θ. To prove the theorem we need to show that

$$\inf_{g\in\mathbb{G}}\limsup_{c\downarrow 0,a\downarrow 0,\ell\uparrow\infty,N\uparrow\infty}\int_0^T ds\sup_f\left\{\int Z^{N,c,a}_{k,\ell,g}(G_s,\eta)f(\eta)\nu^{\alpha,N}_{\gamma_\theta(\cdot)}(d\eta)-\frac{1}{B}N^{2-d}\mathcal{D}^0_N(\sqrt{f},\nu^{\alpha,N}_{\gamma_\theta})\right\}=0$$

for every $B>0$ and then let $B\uparrow\infty$. Notice that for N large enough and a,c small enough, the function $Z^{N,c,a}_{k,\ell,g}(G_s,\eta)$ depends on the configuration η only through the variables $\{\eta(x),\ x\in\Lambda_{(1-\theta)N}\}$. Since $\gamma_\theta(\cdot)$ is constant, say equal to γ_0 in $\Lambda_{(1-\theta)}$, we may replace $\nu^{\alpha,N}_{\gamma_\theta(\cdot)}$ in the previous formula by $\nu^{\alpha,N}_{\gamma_0}$. The $\nu^{\alpha,N}_{\gamma_0}$ is reversible for $\mathcal{L}^0_N$ and therefore $\mathcal{D}^0_N(\cdot,\nu^{\alpha,N}_{\gamma_0})$ is the Dirichlet form associated to the generator $\mathcal{L}^0_N$. Since the Dirichlet form is convex, it remains to show that

$$\inf_{g\in\mathbb{G}}\limsup_{c\downarrow 0,a\downarrow 0,\ell\uparrow\infty,N\uparrow\infty}\int_0^T ds\sup_f\left\{\int Z^{N,c,a}_{k,\ell,g}(G_s,\eta)f(\eta)\nu^{\alpha,N}_{\gamma_0(\cdot)}(d\eta)-\frac{1}{B}N^{2-d}\mathcal{D}^0_N(\sqrt{f},\nu^{\alpha,N}_{\gamma_0})\right\}=0$$

for every $B>0$. This result has been proved in [8], Proposition 4.1. □

Proof of Proposition 3.3 By (3.21), (3.23) and (3.25), applying Theorem 3.10 we obtain (3.4). □

3.4 Proof of Proposition 3.4

For $a>0$, $u\in\Lambda$ denote

$$\iota_a(u)=\frac{1}{|[-a,a]^d\cap\Lambda|}\mathbb{1}_{\{[-a,a]^d\cap\Lambda\}}(u); \tag{3.29}$$

and for $A\subset\Lambda$ define the sets $A^\pm$ as

$$A^+=\{(u_1,\dots,u_d)\in A: u_1>0\},\qquad A^-=\{(u_1,\dots,u_d)\in A:u_1<0\}. \tag{3.30}$$

We define similarly A_N^+ and A_N^- when $A_N \subset \Lambda_N$. Let $G(\cdot,\cdot) \in C^{1,2}([0,T] \times \Lambda)$, $\mu \in D([0,T], \mathcal{M}_1(\Lambda))$ and for $0 < a < c < 1$, define the following functional

$$\begin{aligned}\hat{F}^G_{a,c}\big(\mu(\cdot,\cdot)\big) &= \int_0^T ds \int_{\Lambda_{(1-c)}} du \Big\{ G_s(u)\,(2c)^{-1}\Big[\big(\mu_s \star \iota_a\big)(u+ce_1) - \big(\mu_s \star \iota_a\big)(u-ce_1)\Big]\Big\} \\ &\quad + \int_0^T ds \int_\Lambda du\, \partial_{e_1} G_s(u)\big(\mu_s \star \iota_a\big)(u) - \int_0^T ds \Big\{ \int_\Gamma b(u)\mathbf{n}_1(u) G_s(u) \mathrm{dS}\Big\},\end{aligned} \tag{3.31}$$

where $G_s(u) \equiv G(s,u)$, $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_d)$ is the outward unit normal vector to the boundary surface Γ and dS is the surface element of Γ. The proof of Proposition 3.4 follows from the next lemma.

Lemma 3.11 *For $G(\cdot,\cdot) \in C^{1,2}([0,T] \times \Lambda)$, $\mathbb{P}$-a.s. we have*

$$\limsup_{c\to 0} \limsup_{a \to 0} \limsup_{N\to\infty} \mathbf{E}^{Q_{\mu^N}} \Big[\Big| \hat{F}^G_{a,c}\big(\mu^N(\cdot,\cdot)\big)\Big|\Big] = 0.$$

Proof To short notation, denote $f_s(u) := (\mu_s \star \iota_a\big)(u)$. Taylor expanding we have that

$$\begin{aligned}&\int_{\Lambda_{(1-c)}} du \Big\{ G_s(u)\,(2c)^{-1}\Big[f_s(u+ce_1) - f_s(u-ce_1)\Big]\Big\} \\ &\quad = \frac{1}{2c}\int_{(\Lambda\setminus\Lambda_{(1-2c)})^+} G_s(u-ce_1) f_s(u) du - \frac{1}{2c}\int_{(\Lambda\setminus\Lambda_{(1-2c)})^-} G_s(u+ce_1) f_s(u) du \\ &\qquad - \int_{\Lambda_{(1-2c)}} \partial_{e_1} G_s(u) f_s(u) du + c\int_{\Lambda_{(1-2c)}} R(G,c,s,u) f_s(u) du,\end{aligned} \tag{3.32}$$

where $|R(G,c,s,u)| \le \sup_{u\in\Lambda}\sup_{s\in[0,T]} |\partial^2_{e_1} G_s(\cdot)|$. Since $f_s(u) \le 1$ uniformly in s and u

$$\left| \int_{\Lambda(1-c)} R(G,c,s,u) f_s(u) du \right| \le 2 \sup_{u\in\Lambda}\sup_{s\in[0,T]} |\partial^2_{e_1} G_s(u)|, \tag{3.33}$$

and

$$\left| \int_{\Lambda(1-2c)} \partial_{e_1} G_s(u) f_s(u) du - \int_\Lambda \partial_{e_1} G_s(u) f_s(u) du \right| \le 2c \sup_{u\in\Lambda}\sup_{s\in[0,T]} |\partial_{e_1} G_s(u)|.$$

Taking in account (3.32), (3.29) and (3.33) the lemma is then proven once we show that $\mathbb{P}$-a.s. the following holds

$$\begin{aligned}\limsup_{c\to 0} \limsup_{a \to 0} \limsup_{N\to\infty} \mathbf{E}_{\mu^N} \Bigg[\Bigg| \int_0^T ds \Bigg\{ \frac{1}{2cN^d} \sum_{x \in (\Lambda_{(1-a)N}\setminus\Lambda_{(1-a-2c)N})^\pm} G_s\Big(\frac{x}{N}\Big) \eta_s^{aN}(x) \\ - \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N^\pm} b\Big(\frac{x}{N}\Big) G_s\Big(\frac{x}{N}\Big)\Bigg\}\Bigg|\Bigg] = 0,\end{aligned} \tag{3.34}$$

where for $0 < \varepsilon < 1$, $\Lambda_{\varepsilon N}$ and $(\Lambda_{\varepsilon N})^+$ are defined in (3.24) and below (3.30). By adding and subtracting the same quantity in the expectation of (3.34), it is easy to see that the limit (3.34) follows once the next two lemmas are proven. □

Lemma 3.12 *For $G(\cdot,\cdot) \in C^{1,2}([0,T] \times \Lambda)$, $\mathbb{P}$-a.s. we have*

$$\lim_{\ell\to\infty} \limsup_{c\to 0} \limsup_{a\to 0} \limsup_{N\to\infty} \mathbf{E}_{\mu^N}\Bigg[\Bigg|\int_0^T ds \Bigg\{ \frac{1}{2cN^d} \sum_{x\in(\Lambda_{(1-a)N}\setminus\Lambda_{(1-a-2c)N})^\pm} G_s(x/N)\eta_s^{aN}(x) - \frac{1}{N^{d-1}} \sum_{x\in\Gamma^\pm_{(1-\ell/N)N}} G_s(x/N)\eta_s^\ell(x)\Bigg\}\Bigg|\Bigg] = 0. \tag{3.35}$$

Lemma 3.13 *For $G(\cdot,\cdot) \in C^{1,2}([0,T] \times \Lambda)$, $\mathbb{P}$-a.s. we have*

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \mathbf{E}_{\mu^N}\Bigg[\Bigg|\int_0^T ds \Bigg\{ \frac{1}{N^{d-1}} \sum_{x\in\Gamma^\pm_{(1-\ell/N)N}} G_s(x/N)\eta_s^\ell(x) - \frac{1}{N^{d-1}} \sum_{x\in\Gamma^\pm_N} b(x/N)G_s(x/N)\Bigg\}\Bigg|\Bigg] = 0. \tag{3.36}$$

Proof of Lemma 3.12 The summation in (3.35) contains two similar terms. We consider the one corresponding to the summation of the right-hand side of Λ_N (i.e. the one with signe $+$). By Taylor expansion applied to the function G, the expectation in the statement of the lemma is bounded above by

$$\mathbf{E}_{\mu^N}\Bigg[\Bigg|\int_0^T ds \frac{1}{N^{d-1}} \sum_{\check{x}\in\mathbf{T}_N^{d-1}} G_s\Big(1, \frac{\check{x}}{N}\Big) \Bigg\{ \frac{1}{2cN} \sum_{x_1=N(1-a-2c)+1}^{N(1-a)} \Big(\eta_s^{aN}(x_1,\check{x}) - \eta_s^\ell(N-\ell,\check{x})\Big)\Bigg\}\Bigg|\Bigg] + R(N,a,c,G),$$

where for $x_1 \in [-N,N]$, $\check{x} = (x_2,\dots,x_d) \in \mathbf{T}_N^{d-1}$ the vector $(x_1,\check{x})$ stands for the element $(x_1,x_2,\dots,x_d) \in \Lambda_N$. We denoted by $R(N,a,c,G)$ a quantity so that for $G \in C^{1,2}([0,T]\times\Lambda)$,

$$\limsup_{c\to 0} \limsup_{a\to 0} \limsup_{N\to\infty} |R(N,a,c,G)| = 0. \tag{3.37}$$

The next step consists in replacing the density average over a small macroscopic box of length aN by a large microscopic box. More precisely, for N large enough, the expectation of the last quantity is bounded above by

$$C\|G\|_\infty \sup_{2\ell<|y|\le 2Nc} \mathbf{E}_{\mu^N}\Bigg[\int_0^T ds \frac{1}{N^{d-1}} \sum_{\check{x}\in\mathbf{T}_N^{d-1}} \Big|\eta_s^\ell\big((N-\ell,\check{x})+y\big) - \eta_s^\ell(N-\ell,\check{x})\Big|\Bigg] + R(N,a,c,\ell), \tag{3.38}$$

where for all ℓ, $R(N,a,c,\ell)$ satisfy (3.37) and C is a positive constant. Observe that the first term of the previous formula is not depending on a but only on c, N and ℓ.

In view of the estimate (3.12) and Lemma 3.8, by the usual two blocks estimate, the first term of (3.38) converges to 0 an $N \uparrow \infty$, $c \downarrow 0$ and $\ell \uparrow \infty$. This concludes the proof of Lemma 3.12. □

Proof of Lemma 3.13 The summation in (3.36) contains two similar terms. We consider the one corresponding to the summation of the right-hand side of Λ_N. Taylor expanding $G_s(\cdot)$ we bound above the expectation in (3.36) by

$$\|G\|_\infty \frac{1}{N^{d-1}} \sum_{y\in\Gamma_N^+} \mathbf{E}_{\mu^N}\left[\int_0^T ds \left|\eta_s^\ell(y-\ell e_1) - b(y/N)\right|\right] + C\frac{\ell}{N}, \tag{3.39}$$

where C is a positive constant depending on T and $\|\nabla G\|_\infty$. For any fixed positive integer ℓ denote by $\Gamma_0^\ell = \{(0,\hat{x}) : \hat{x} \in \mathbf{T}_N^{d-1},\ |\hat{x}| \le \ell\} = (\{0\} \times \mathbf{T}_N^{d-1}) \cap \Lambda_\ell(0)$, for notation see (3.18). For $u \in \Gamma$, denote

$$\tilde{D}_{\ell,0}^{b,u}(f,\nu) = \frac{1}{2}\sum_{x\in\Gamma_0^\ell} \int \tilde{C}_0^b(u,x,\eta)\big(f(\eta^x) - f(\eta)\big)^2 d\nu(\eta),$$

where

$$\tilde{C}_0^b(u,x,\eta) = \eta(x)\exp\left\{-\frac{\alpha(x)+\lambda_0(b(u))}{2}\right\} + (1-\eta(x))\exp\left\{\frac{\alpha(x)+\lambda_0(b(u))}{2}\right\}. \tag{3.40}$$

The difference with the rate in (2.11) is that here u is fixed.

Let $\nu_{b(u)}^{\alpha,N}$ be the product measure with constant profile $b(u)$. Let $f : \mathcal{S}_N \to \mathbb{R}$, denote by f^ℓ the conditional expectation of f with respect to the σ-algebra generated by $\{\eta(z) : z \in \Lambda_\ell(0)\}$:

$$f^\ell(\xi) = \frac{1}{\nu_{b(u)}^{\alpha,\ell}(\xi)} \int \mathbb{1}_{\{\eta;\eta(z)=\xi(z),z\in\Lambda_\ell(0)\}} f(\eta)\, d\nu_{b(u)}^{\alpha,N}(\eta) \quad \text{for all } \xi \in \{0,1\}^{\Lambda_\ell(0)},$$

where $\nu_{b(u)}^{\alpha,\ell}$ is the restriction of $\nu_{b(u)}^{\alpha,N}$ to $\{0,1\}^{\Lambda_\ell(0)}$.

Note that $|\eta^\ell(0) - b(u)|$ depends only on coordinates on the box $\Lambda_\ell(0)$, then by Fubini's Theorem,

$$\mathbf{E}_{\mu^N}\left[\int_0^T ds\left|\eta_s^\ell(y-\ell e_1) - b(y/N)\right|\right] = T\int \left|\eta^\ell(0) - b(y/N)\right| \big(\tau_{-(y-\ell e_1)}\bar{f}_T^{y,N}\big)^\ell(\eta)\, d\nu_{b(\frac{y}{N})}^{\alpha,\ell}(\eta) \tag{3.41}$$

where $\bar{f}_T^{y,N} = \frac{1}{T}\int_0^T f_s^{y,N} ds$ and for all $0 \le s \le T$, $f_s^{y,N}$ is the density of μ_s^N with respect to the product measure $\nu_{b(\frac{y}{N})}^{\alpha,N}$ with constant profile $b(\frac{y}{N})$. The density $\big(\tau_{-(y-\ell e_1)}\bar{f}_T^{y,N}\big)^\ell$ stands for the conditional expectation of $\tau_{-(y-\ell e_1)}\bar{f}_T^{y,N}$ with respect to the σ-algebra generated by $\{\eta(z) : z \in \Lambda_\ell(0)\}$.

Remark that, since the Dirichlet form is convex and since the conditional expectation is an average,

$$\begin{aligned}
\tilde{D}_{\ell,0}^{b,\frac{y}{N}}\left(\sqrt{\big(\tau_{-(y-\ell e_1)}\bar{f}_T^{y,N}\big)^\ell}, \nu_{b(y/N)}^{\alpha,\ell}\right) &\le \tilde{D}_{\ell,0}^{b,\frac{y}{N}}\left(\sqrt{\tau_{-(y-\ell e_1)}\bar{f}_T^{y,N}}, \nu_{b(y/N)}^{\alpha,N}\right) \\
&= \mathcal{D}_{\ell,y-\ell e_1}^b\left(\sqrt{\bar{f}_T^{y,N}}, \nu_{b(y/N)}^{\alpha,N}\right) \\
&\le \frac{1}{T}\int_0^T \mathcal{D}_{\ell,y-\ell e_1}^b\left(\sqrt{f_s^{y,N}}, \nu_{b(y/N)}^{\alpha,N}\right) ds. \qquad (3.42)
\end{aligned}$$

Applying Lemma 3.9 we obtain from (3.42)

$$
\begin{aligned}
&N^{1-d}\sum_{y\in\Gamma_N}\tilde{D}^{b,\frac{y}{N}}_{\ell,0}\Big(\sqrt{\big(\tau_{-(y-\ell e_1)}\bar{f}^{y,N}_T\big)^{\ell}},\nu^{\alpha,\ell}_{b(y/N)}\Big)\\
&\quad\le\frac{1}{T}\int_0^T\Big\{N^{1-d}\sum_{y\in\Gamma_N}\mathcal{D}^b_{\ell,y-\ell e_1}\Big(\sqrt{f^{y,N}_s},\nu^{\alpha,N}_{b(y/N)}\Big)\Big\}ds\\
&\quad\le 2\frac{1}{T}\int_0^T\Big\{N^{1-d}\sum_{y\in\Gamma_N}\mathcal{D}^b_{\ell,y-\ell e_1}\Big(\sqrt{h^N_s},\nu^{\alpha,N}_{\gamma(\cdot)}\Big)\Big\}ds+C_0\frac{\ell^{d+1}}{N^2}\\
&\quad\le\frac{C_T}{N}+C_0\frac{\ell^{d+1}}{N^2},
\end{aligned}
\tag{3.43}
$$

for some constant C_T that depends on T. By the same argument we obtain the bound on the Dirichlet form $\mathcal{D}^0_{\ell,0}$,

$$
N^{1-d}\sum_{y\in\Gamma_N}\mathcal{D}^0_{\ell,0}\Big(\sqrt{\big(\tau_{-(y-\ell e_1)}\bar{f}^{y,N}_T\big)^{\ell}},\nu^{\alpha,\ell}_{b(y/N)}\Big)\le\frac{C_T}{N}+C_0\frac{\ell^d}{N^2}.
\tag{3.44}
$$

Therefore, for N large enough, for all positive integer $k\ge 1$ we can bound the expectation in (3.39) as following

$$
\begin{aligned}
&\frac{1}{N^{d-1}}\sum_{y\in\Gamma^+_N}\mathbf{E}_{\mu^N}\Big[\int_0^T ds\Big|\eta^{\ell}_s(y-\ell e_1)-b(y/N)\Big|\Big]\\
&\le\frac{1}{N^{d-1}}\sum_{y\in\Gamma^+_N}\Big\{\int\Big|\eta^{\ell}(0)-b(y/N)\Big|\big(\tau_{-(y-\ell e_1)}\bar{f}^{y,N}_T\big)^{\ell}d\nu^{\alpha,\ell}_{b(y/N)}(\eta)\\
&\quad-k\mathcal{D}^0_{\ell,0}\Big(\sqrt{\big(\tau_{-(y-\ell e_1)}\bar{f}^{y,N}_T\big)^{\ell}},\nu^{\alpha,\ell}_{b(y/N)}\Big)-k\tilde{D}^{b,\frac{y}{N}}_{\ell,0}\Big(\sqrt{\big(\tau_{-(y-\ell e_1)}\bar{f}^{y,N}_T\big)^{\ell}},\nu^{\alpha,\ell}_{b(y/N)}\Big)\Big\}\\
&\quad+2\frac{k}{N}\Big(C_T+C_0\frac{\ell^d(\ell+1)}{N}\Big).
\end{aligned}
$$

This last expression is bounded above by

$$
\begin{aligned}
&\frac{1}{N^{d-1}}\sum_{y\in\Gamma^+_N}\sup_{f\in\mathcal{A}^{\frac{y}{N}}_{\ell}}\Big\{\int\Big|\eta^{\ell}(0)-b(y/N)\Big|f(\eta)d\nu^{\alpha,\ell}_{b(y/N)}(\eta)-k\mathcal{D}^0_{\ell,0}\Big(\sqrt{f},\nu^{\alpha,\ell}_{b(y/N)}\Big)\\
&\quad-k\tilde{D}^{b,\frac{y}{N}}_{\ell,0}\Big(\sqrt{f},\nu^{\alpha,\ell}_{b(y/N)}\Big)\Big\}+2\frac{k}{N}\Big(C_T+C_0\frac{\ell^d(\ell+1)}{N}\Big),
\end{aligned}
\tag{3.45}
$$

where, for $u\in\Gamma$,

$$
\mathcal{A}^u_{\ell}=\Big\{f:f\ge 0,\int f(\xi)d\nu^{\alpha,\ell}_{b(u)}(\xi)=1\Big\}.
$$

Further, since the function

$$
u\to\sup_{f\in\mathcal{A}^u_{\ell}}\Big\{\int\Big|\eta^{\ell}(0)-b(u)\Big|f(\eta)d\nu^{\alpha,\ell}_{b(u)}(\eta)-k\,\mathcal{D}^0_{\ell,0}\Big(\sqrt{f},\nu^{\alpha,\ell}_{b(u)}\Big)-k\tilde{D}^{b,u}_{\ell,0}\Big(\sqrt{f},\nu^{\alpha,\ell}_{b(u)}\Big)\Big\}
$$

is continuous on Γ, from Lemma 3.5, for all positive integers ℓ and k, the limit when $N \uparrow \infty$ of the expression (3.45) is equal to

$$\int_{\Gamma^+} du\, \mathbb{E}\Big[\sup_{f\in\mathcal{A}_\ell^u}\Big\{\int \Big|\eta^\ell(0)-b(u)\Big| f d\nu_{b(u)}^{\alpha,\ell}(\eta) - k\,\mathcal{D}_{\ell,0}^0\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big) - k\tilde{D}_{\ell,0}^{b,u}\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big)\Big\}\Big].$$

Since $\int |\eta^\ell(0)-b(u)| f d\nu_{b(u)}^{\alpha,\ell}(\eta) \le C_b$ for some positive constant C_b that depends on $\|b\|_\infty$, the integral on Γ^+ in the last expression is bounded by

$$\int_{\Gamma^+} du \mathbb{E}\Big[\sup_{f\in\mathcal{A}_{\ell,k,C_b}^u}\Big\{\int \Big|\eta^\ell(0)-b(u)\Big| f(\eta) d\nu_{b(u)}^{\alpha,\ell}(\eta)\Big\}\Big],$$

where for a positive constant C, $\mathcal{A}_{\ell,k,C}^u$ is the following set of densities,

$$\mathcal{A}_{\ell,k,C}^u = \Big\{f\in\mathcal{A}_\ell^u,\, \tilde{D}_{\ell,0}^{b,u}\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big)\le\frac{C}{k},\, \mathcal{D}_{\ell,0}^0\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big)\le\frac{C}{k}\Big\}.$$

We first consider the limit when $k \uparrow \infty$ and use the usual technique in the replacement lemma. Since for any $\ell > 1$, any constant $C > 0$ and any $u \in \Gamma$ the sets $\mathcal{A}_{\ell,k,C}^u$ are compacts for the weak topology, for all $\ell > 1$

$$\begin{aligned}\limsup_{k\to\infty} \sup_{f\in\mathcal{A}_{\ell,k,C}^u}&\Big\{\int \Big|\eta^\ell(0)-b(u)\Big| f(\eta) d\nu_{b(u)}^{\alpha,\ell}(\eta)\Big\}\\ &= \sup_{f\in\mathcal{A}_{\ell,0}^u}\Big\{\int \Big|\eta^\ell(0)-b(u)\Big| f(\eta) d\nu_{b(u)}^{\alpha,\ell}(\eta)\Big\},\end{aligned}$$

where

$$\mathcal{A}_{\ell,0}^u = \Big\{f\in\mathcal{A}_\ell^u,\, \tilde{D}_{\ell,0}^{b,u}\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big)=0,\, \mathcal{D}_{\ell,0}^0\Big(\sqrt{f},\nu_{b(u)}^{\alpha,\ell}\Big)=0\Big\}.$$

By dominated convergence theorem, it is then enough to show that,

$$\limsup_{\ell\to\infty} \mathbb{E}\Big[\sup_{f\in\mathcal{A}_{\ell,0}^u}\Big\{\int \Big|\eta^\ell(0)-b(u)\Big| f(\eta) d\nu_{b(u)}^{\alpha,\ell}(\eta)\Big\}\Big] = 0.$$

Now, it is easy to see that, due to the presence of the jumps of particles in the Dirichlet form $\mathcal{D}_{\ell,0}^0$ and the presence of the creation and destruction of particles in $\tilde{D}_{\ell,0}^{b,u}$ the set $\mathcal{A}_{\ell,0}^u = \{1\}$. Thus, to conclude the proof of the lemma, it remains to apply the usual law of large numbers. □

Proof of Proposition 3.4 Let Q^* be a limit point of the sequence $(Q_{\mu^N})_{N\ge1}$ and let $(Q_{\mu^{N_k}})_{k\ge1}$ be a sub-sequence converging to Q^*. By Lemma 3.2, Q^* is concentrated on the trajectories that are in $L^2([0,T];H^1(\Lambda))$. For $0<c<1$ and for $\mu(\cdot,\cdot)\in D([0,T],\mathcal{M}_1^0(\Lambda))$, such that $\mu(t,du)=\rho(t,u)du$ with $\rho(\cdot,\cdot)\in L^2([0,T];H^1(\Lambda))$, denote by $F_c^G(\mu)$ the functional

$$\begin{aligned}F_c^G\big(\mu(\cdot,\cdot)\big) &= \int_0^T ds \int_{\Lambda_{(1-c)}} du\Big\{G_s(u)\,(2c)^{-1}\Big[\rho(s,u+ce_1)-\rho(s,u-ce_1)\Big]\Big\}\\ &\quad + \int_0^T ds\int_\Lambda du\partial_{e_1}G_s(u)\rho(s,u) - \int_0^T ds\Big\{\int_\Gamma b(u)\mathbf{n}_1(u)G_s(u)\mathrm{dS}\Big\}.\end{aligned}$$

From Lemma 3.11 and the continuity of the function $\mu \to \hat{F}^G_{a,c}(\mu)$, we have

$$\limsup_{c\to 0} E^{Q^*}\Big[\Big|F^G_c(\mu)\Big|\Big] = 0. \tag{3.46}$$

On the other hand, an integration by parts and Taylor expansion up to the second order of the function $G_s(\cdot)$ permit to rewrite F^G_c as

$$\begin{aligned} F^G_c\big(\mu(\cdot,\cdot)\big) &= \int_0^T \frac{1}{2c}\int_{(\Lambda\setminus\Lambda_{(1-2c)})^+} G_s(u)\rho(s,u)du ds \\ &\quad - \int_0^T \frac{1}{2c}\int_{(\Lambda\setminus\Lambda_{(1-2c)})^-} G_s(u)\rho(s,u)du ds \\ &\quad - \int_0^T ds\int_\Gamma b(u)\mathbf{n}_1(u)G_s(u)\mathrm{dS} + R(c), \end{aligned}$$

where $R(c)\equiv R(G,c)$ is a function vanishing as $c\downarrow 0$. Further one has, see Theorem 5.3.2. of [5], that

$$\lim_{r\to 0}\frac{1}{|B(u,r)\cap\Lambda|}\int_{B(u,r)\cap\Lambda}\rho(s,y)dy = \mathrm{Tr}(\rho(s,u)) \quad \text{a.e } u\in\Gamma,\ \forall s\in[0,T], \tag{3.47}$$

and then by dominated convergence theorem

$$\lim_{c\to 0} F^G_c\big(\mu(\cdot,\cdot)\big) = \int_0^T ds\int_\Gamma (\mathrm{Tr}(\rho(s,u)) - b(u))\mathbf{n}_1(u)G_s(u)\mathrm{dS}.$$

This, together with (3.46) implies

$$E^{Q^*}\Big[\Big|\int_0^T ds\int_\Gamma(\mathrm{Tr}(\rho(s,u)) - b(u))\mathbf{n}_1(u)G_s(u)\mathrm{dS}\Big|\Big] = 0,$$

which concludes the proof. □

4 Proof of Theorem 2.2

Denote by $Q^N_s := Q^{N,\alpha}_{\nu_s{}^{\alpha,N}}$ the probability measure on the Skorohod space $D\big([0,T],\mathcal{M}\big)$ induced by the Markov process $(\pi^N_t)\equiv(\pi_N(\eta_t))$, when the initial measure is $\nu_s{}^{\alpha,N}$. The main problem in proving Theorem 2.2 is that we do not know that the empirical initial measure at time zero converges to a macroscopic profile according to definition (2.23). If this would be the case the result would be a corollary of Theorem 2.1. Taking this in account we first prove that Q^N_s is a tight sequence and that all its limit points are concentrated on weak solution of the hydrodynamic equation for some unknown initial profile. This is in contrast with the usual hydrodynamic limit, in which one associates the initial empirical measure to a profile. Then we show the uniqueness and the atractivity of the stationary solution of (2.21) for the evolution (2.15) to conclude.

Denote by $\mathcal{A}_T\subset D([0,T],\mathcal{M})$ the class of profiles $\rho(\cdot,\cdot)$ that satisfies conditions (IB1), (IB2) and (IB3). The first step to show Theorem 2.2 consists in proving that all limit points of the sequence (Q^N_s) are concentrated on $\mathcal{A}_T$:

Proposition 4.1 *The sequence of probability measures* (Q_s^N) *is weakly relatively compact and all its converging subsequences converge to the some limit* Q_s^* *that is concentrated on the absolutely continuous measures* $\pi(t,du)=\rho(t,u)du$ *whose density* ρ *satisfying* (IB1), (IB2) *and* (IB3).

The proof of Proposition 4.1 follows the same steps needed to show Theorem 2.1. We just have to show the analogous of Lemmas 3.6–3.9 when the measure μ^N in the statements of these lemmas is replaced by ${\nu_s}^{\alpha,N}$. The only lemma to be slightly modified is Lemma 3.8, see Lemma 4.2 given next. Recall that $\gamma:\Lambda\to(0,1)$ is a smooth profile equal to b at the boundary of Λ. Let h^N be the density of ${\nu_s}^{\alpha,N}$ with respect to the measure $\nu^{\alpha,N}_{\gamma(\cdot)}$.

Lemma 4.2 *There exists positive constant* $C=C(\|\nabla\gamma\|_\infty)$ *depending only on* $\gamma(\cdot)$ *such that for any* $a>0$

$$(1-a)\mathcal{D}_N^0(\sqrt{h^N},\nu^{\alpha,N}_{\gamma(\cdot)})+\mathcal{D}_N^b(\sqrt{h^N},\nu^{\alpha,N}_{\gamma(\cdot)})\le\frac{C}{a}N^{d-2}.$$

Proof By the stationary of ${\nu_s}^{\alpha,N}$,

$$\partial_t H_N(t)=\int_{\mathcal{S}_N}h^N\mathcal{L}_N\log\big(h^N\big)d\nu^{\alpha,N}_{\gamma(\cdot)}=0.$$

Recalling that the generator $\mathcal{L}_N$ has two pieces and applying the inequality $a(\log b-\log a)\le-(\sqrt{a}-\sqrt{b})^2+(b-a)$ for positive a and b, we obtain

$$\begin{aligned}0&=\int_{\mathcal{S}_N}h^N\mathcal{L}_N\log\big(h^N\big)d\nu^{\alpha,N}_{\gamma(\cdot)}\\&\le-2N^2\mathcal{D}_N^0\big(\sqrt{h^N},\nu^{\alpha,N}_{\gamma(\cdot)}\big)-2N^2\mathcal{D}_N^b\big(\sqrt{h^N},\nu^{\alpha,N}_{\gamma(\cdot)}\big)\\&\quad+N^2\int_{\mathcal{S}_N}\mathcal{L}_N^0h^Nd\nu^{\alpha,N}_{\gamma(\cdot)}+N^2\int_{\mathcal{S}_N}\mathcal{L}_N^bh^Nd\nu^{\alpha,N}_{\gamma(\cdot)}.\end{aligned}$$

We then apply the same computation as in the proof of Lemma 3.8 ((3.15) and (3.16)). □

Proof of Theorem 2.2 Let Q_s^* be a limit point of (Q_s^N) and $(Q_s^{N_k})$ be a sub-sequence converging to Q_s^*. Let $\bar\rho$ be the stationary solution of (2.15), see (2.21). We have by Proposition 4.1 the following:

$$\begin{aligned}\lim_{k\to\infty}Q_s^{N_k}\Big\{\Big|\langle\pi_T^N,G\rangle-\langle\overline{\rho}(u)du,G\rangle\Big|\Big\}&=Q_s^*\Big\{\Big|\langle\rho(T,\cdot),G\rangle-\langle\overline{\rho}(u)du,G\rangle\Big|\mathbb{1}\{\mathcal{A}_T\}\big(\rho\big)\Big\}\\&\le\|G\|_\infty Q_s^*\Big\{\big\|\rho(T,\cdot)-\overline{\rho}(\cdot)\big\|_1\mathbb{1}\{\mathcal{A}_T\}\big(\rho\big)\Big\},\end{aligned}$$

where $\|v\|_1$ denotes the $L^1(\Lambda)$ norm of v. By the stationary of ${\nu_s}^{\alpha,N}$

$$Q_s^{N_k}\Big\{\Big|\langle\pi_T^N,G\rangle-\langle\overline{\rho}(u)du,G\rangle\Big|\Big\}=Q_s^{N_k}\Big\{\Big|\langle\pi^N,G\rangle-\langle\overline{\rho}(u)du,G\rangle\Big|\Big\}.$$

Denote by $\rho^0(\cdot,\cdot)$ (resp. $\rho^1(\cdot,\cdot)$) the element of $\mathcal{A}_T$ with initial condition $\rho^0(0,\cdot)\equiv0$ (resp. $\rho^1(0,\cdot)\equiv1$). From Lemma A.7, each profile $\rho(\cdot,\cdot)\in\mathcal{A}_T$ is such that for all $t\ge0$, $\lambda\{u\in\Lambda:$

$0 \le \rho^0(t,u) \le \rho(t,u) \le \rho^1(t,u) \le 1\} = 1$ and $\lambda\{u \in \Lambda : \rho^0(t,u) \le \overline{\rho}(u) \le \rho^1(t,u)\} = 1$, where λ is the Lebesgue measure on Λ. Therefore

$$\lim_{k\to\infty} Q_s^{N_k}\left\{\left|\langle \pi^N, G\rangle - \langle \overline{\rho}(u)du, G\rangle\right|\right\} \le \|G\|_\infty \left\|\rho^0(T,\cdot) - \rho^1(T,\cdot)\right\|_1, \quad \mathbb{P}\text{-a.s.}$$

Note that the left hand side does not depend on T. To conclude the proof, it is enough to let $T \uparrow \infty$ and to apply Theorem A.10. □

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Appendix A

In this section we prove the existence and uniqueness of the weak solution of (2.15) and (2.21). Further we show, in Theorem A.10 the global stability of the stationary solution of (2.15). The proof of these results is based on an extensive use of monotone methods, see [21]. We were not able to find the precise reference in the literature, so we briefly sketch them for completeness.

We need to introduce some extra notation. Let $\mathcal{C}^{1,2}([0,T]\times\Lambda)$ be the space of functions from $[0,T]\times\Lambda$ to $\mathbb{R}$ twice continuously differentiable in Λ with continuous time derivative. Denote by

$$\begin{aligned}\mathcal{G} := \{G \in \mathcal{C}^{1,2}([0,T]\times\Lambda), G(t,u) = G_t(u) \text{ pointwise positive},\\ G(t,u) = 0,\ \forall u \in \Gamma,\ \forall t \in [0,T]\}.\end{aligned}$$

It is convenient to reformulate the notion of weak solution of (2.15) as following. A function $\rho(\cdot,\cdot) : [0,T]\times\Lambda \to [0,1]$ is a weak solution of the initial-boundary value problem (2.15) if $\rho \in L^2(0,T; H^1(\Lambda))$ and for every $G \in \mathcal{G}$

$$\begin{aligned}&\int_\Lambda du\{G_T(u)\rho(T,u) - G_0(u)\rho_0(u)\} - \int_0^T ds \int_\Lambda du\,(\partial_s G_s)(u)\rho(s,u)\\ &\quad = \sum_{i,j}\int_0^T ds\left\{\int_\Lambda du\, A_{i,j}(\rho(s,u))\frac{\partial^2}{\partial_{i,j}}G_s(u) - \int_\Gamma A_{i,j}(b(u))\partial_{n_1}G(s,u)dS\right\} \qquad \text{(A.1)}\end{aligned}$$

where $A_{i,j}(\rho) = \int_0^\rho D_{i,j}(\rho')d\rho'$. A function $\rho^+(\cdot,\cdot) : [0,T]\times\Lambda \to \mathbb{R}$ is a weak *upper solution* of the initial-boundary value problem (2.15) if $\rho^+ \in L^2\big(0,T; H^1(\Lambda)\big)$ and for all $G \in \mathcal{G}$ we have

$$\begin{cases}\displaystyle\sum_{i,j}\int_0^T ds\left\{\int_\Lambda du\, A_{i,j}(\rho^+(s,u))\frac{\partial^2}{\partial_{i,j}}G_s(u) - \int_\Gamma A_{i,j}(\rho^+(s,u))\partial_{n_1}G(s,u)dS\right\}\\ \displaystyle\quad - \int_\Lambda du\{G_T(u)\rho^+(T,u) - G_0(u)\rho_0^+(u)\} - \int_0^T ds\int_\Lambda du\,(\partial_s G_s)(u)\rho^+(s,u) \le 0,\\ \mathrm{Tr}(\rho^+(t,\cdot)) \ge b(\cdot) \quad \text{on } \Gamma\\ \rho^+(0,u) \ge \rho_0(u) \quad u \in \Lambda.\end{cases} \qquad \text{(A.2)}$$

A weak *lower solution* $\rho^-(\cdot,\cdot):[0,T]\times\Lambda\to\mathbb{R}$ is defined reversing the inequalities in (A.2). By a solution of the stationary problem (2.15) we mean a function $\bar\rho\in H^1(\Lambda)$ so that for all $G\in\mathcal{C}^2(\Lambda)$, *pointwise positive* vanishing on Γ

$$\sum_{i,j}\left\{\int_\Lambda du\, A_{i,j}(\bar\rho(u)))\frac{\partial^2}{\partial_{i,j}}G(u)-\int_\Gamma A_{i,j}(b(u))\partial_{n_1}G(u)dS\right\}=0 \tag{A.3}$$

As before we define upper and lower solutions of the stationary problem (A.3). A function $\bar\rho^+$ is an upper solution for the stationary problem (A.3) if $\bar\rho^+\in H^1(\Lambda)$ and for all $G\in\mathcal{C}^2(\Lambda)$, pointwise positive vanishing on Γ,

$$\begin{cases}\displaystyle\sum_{i,j}\left\{\int_\Lambda du\, A_{i,j}(\bar\rho^+(u)))\frac{\partial^2}{\partial_{i,j}}G(u)-\int_\Gamma A_{i,j}(\bar\rho^+(u))\partial_{n_1}G(u)dS\right\}\le 0,\\ \mathrm{Tr}(\bar\rho^+)\ge b\quad \text{on }\Gamma.\end{cases} \tag{A.4}$$

A lower solution of the stationary problem (A.3) is defined reversing the inequality in (A.4).

Denote by $H^{-1}(\Lambda)$ the dual of $H_0^1(\Lambda)$, i.e. the Banach space equipped with the norm

$$\|v\|_{-1}=\sup_f\left\{\langle v,f\rangle:\|f\|_{H_0^1(\Lambda)}\le 1\right\}. \tag{A.5}$$

To apply the monotone method we first show the following comparison principle.

Lemma A.1 *Let* ρ^1 (*resp.* ρ^2) *be a lower solution* (*resp. upper solution*) *of* (2.15), $\partial_t\rho^i\in L^2(0,T;H^{-1}(\Lambda))$, *for* $i=1,2$. *If there exists* $s\ge 0$ *such that*

$$\lambda\left\{u\in\Lambda:\rho^1(s,u)\le\rho^2(s,u)\right\}=1,$$

where λ *is the Lebesgue measure on* Λ, *then for all* $t\ge s$

$$\lambda\left\{u\in\Lambda:\rho^1(t,u)\le\rho^2(t,u)\right\}=1.$$

Proof Take $s<t<T$ and $\delta>0$. Denote by F_δ the function defined by

$$F_\delta(a):=\frac{a^2}{2\delta}\mathbb{1}_{\{0\le a\le\delta\}}+\big(a-\delta/2\big)\mathbb{1}_{\{a>\delta\}},\quad a\in\mathbb{R}.$$

Let $A_\delta:=A_\delta(T)$ be the set

$$A_\delta=\left\{(t,u)\in[0,T]\times\Lambda:0\le\rho^1(t,u)-\rho^2(t,u)\le\delta\right\}.$$

By definition $\mathrm{Tr}(\rho^1-\rho^2)\le 0$ a.e. and therefore $\mathrm{Tr}(F'_\delta(\rho^1-\rho^2))=0$. Since ρ^1 (ρ^2) is lower (upper) solution of (2.15), we have that

$$\begin{aligned}&\int_s^t d\tau\frac{\partial}{\partial\tau}\int_\Lambda F_\delta(\rho^1(\tau,u)-\rho^2(\tau,u))\\ &\quad=\int_\Lambda du\,F_\delta(\rho^1(t,u)-\rho^2(t,u))-\int_\Lambda du\,F_\delta(\rho^1(s,u)-\rho^2(s,u))\\ &\quad\le-\delta^{-1}\int_s^t d\tau\int_{A_\delta}du\,\nabla(\rho^1-\rho^2)\cdot\{D(\rho^1)\nabla\rho^1-D(\rho^2)\nabla\rho^2\}\end{aligned}$$

$$= -\delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \nabla(\rho^1 - \rho^2) \cdot D(\rho^1) \nabla(\rho^1 - \rho^2)$$
$$- \delta^{-1} \int_s^t d\tau \int_{A_\delta} du\, \nabla(\rho^1 - \rho^2) \cdot \big\{D(\rho^1) - D(\rho^2)\big\} \nabla \rho^2. \tag{A.6}$$

Since $D(\cdot)$ is strictly positive, see (2.19), the third line of (A.6) can be estimated by above

$$-\frac{1}{\delta} \int_s^t d\tau \int_{A_\delta} du\, \nabla(\rho^1 - \rho^2) \cdot D(\rho^1) \nabla(\rho^1 - \rho^2) \le -\frac{1}{\delta C} \int_s^t d\tau \int_{A_\delta} du\, \|\nabla(\rho^1 - \rho^2)\|^2. \tag{A.7}$$

Further, by the Lipschitz property of $D(\cdot)$ we have on the set A_δ, $\sup_{1\le i,j\le d} |D_{i,j}(\rho^1) - D_{i,j}(\rho^2)| \le M|\rho^1 - \rho^2| \le M\delta$ for some positive constant M. By Schwarz inequality, the last line of (A.6) is bounded by

$$\delta^{-1} M A \int_s^t d\tau \int_{A_\delta} du\, \|\nabla(\rho^1 - \rho^2)\|^2 + \delta M A^{-1} \int_s^t d\tau \int_{A_\delta} du\, \|\nabla \rho^2\|^2 \tag{A.8}$$

for every $A > 0$. By (A.6), (A.7), (A.8) and choosing $A = M^{-1}C^{-1}$ to cancel the term in (A.7) and the first term of (A.8) we have

$$\int_\Lambda du\, F_\delta(\rho^1(t,u) - \rho^2(t,u)) - \int_\Lambda du\, F_\delta(\rho^1(s,u) - \rho^2(s,u))$$
$$\le \delta C^{-1} M^2 \int_0^T d\tau \int du\, \|\nabla \rho^2\|^2.$$

Letting $\delta \downarrow 0$, we conclude the proof of the lemma because $F_\delta(\cdot)$ converges to the function $F(a) = a\mathbb{1}_{a\ge 0}$ as $\delta \downarrow 0$. □

We immediately obtain the following.

Proposition A.2 *Let $m_0 : \Lambda \to [0,1]$ be a measurable function. There is a unique weak solution $\rho(t, m_0)$ of* (2.15) *with initial datum m_0.*

Proof Existence of weak solution of (2.15) can be deduced by the tightness of the sequence $Q^N_{\mu_N}$ where μ_N is the probability measure associated to the initial profile m_0 according to (2.23). Uniqueness is a consequence of Lemma A.1. □

Corollary A.3 *Let m_0 be a lower stationary solution of* (A.3). *Let $\rho(t, m_0)$ be the solution of* (A.1) *with initial datum m_0 then $\rho(t,u) \ge m_0(u)$ a.e. in (u,t).*

The proof is an immediate consequence of Lemma A.1 with $\rho^1 := m_0$ and $\rho^2 := \rho$. When the initial datum of solution of (A.1) is an upper stationary solution we have:

Corollary A.4 *Let m_1 be a upper stationary solution of* (A.3). *Let $\rho(t, m_1)$ be the solution of* (A.1) *with initial datum m_1 then $\rho(t,u) \le m_1(u)$ for $t \in [0,T]$ and $u \in \Lambda$.*

Next we show that when a lower (upper) stationary solution m_0 (m_1) is taken as initial datum, the corresponding solution $\rho(t, m_0)$ ($\rho(t, m_1)$) is monotone nondecreasing (nonincreasing) in time.

Lemma A.5 *Under the assumptions of Corollary A.3 $\rho(t,m_0)$ is a nondecreasing solution of (2.15) on $[0,T]$.*

Proof Corollary A.3 implies that $\rho(s,m_0)\ge m_0$ for all $s\ge 0$, since m_0 lower solution. Let $\rho(t;\rho(s,m_0))$ be the solution of (A.1) starting at time $t=0$ from $\rho(s,m_0)$. Then $\rho(t;\rho(s,m_0))\ge \rho(t,m_0)$ since the initial datum $\rho(s,m_0)\ge m_0$. But $\rho(t;\rho(s,m_0))=\rho(t+s,m_0)$ by uniqueness of weak solution then $\rho(t+s,m_0)\ge\rho(t,m_0)\ge m_0$. □

Lemma A.6 *Under the assumptions of Corollary A.4 $\rho(t,m_1)$ is a nonincreasing solution of (2.15) for $t\in[0,T]$.*

The proof is similar to the one of Lemma A.5.

Lemma A.7 *Let m_0 be a lower solution and m_1 be an upper solution of (A.3), $m_0(\cdot)\le m_1(\cdot)$ a.e. in Λ, we have*

$$m_0\le\rho(t;m_0)\le\rho(t;m_1)\le m_1\quad \forall t\in(0,\infty).$$

The proof is an immediate consequence of the previous results.

Lemma A.8 *Under the assumption of Lemma A.7 the solutions $\rho(t;m_0)$ and $\rho(t;m_1)$ exist for all $t\in[0,\infty)$ and they converge in $L^p(\Lambda)$ for $p\in[1,\infty)$ to limits $\rho_\star(\cdot)$ and $\rho^\star(\cdot)$, both solutions of (A.3). Further*

$$\rho_\star(u)\le\rho^\star(u)\quad a.e.$$

Proof Since $\rho(t;m_0)$ is nondecreasing in t and $\rho(t;m_0)\le m_1$ for any $t\ge 0$, $\rho(t;m_0)$ converges almost everywhere in Λ as $t\to\infty$ and $\rho_\star(\cdot)\in L^\infty(\Lambda)$. By the monotone convergence theorem $\rho(t;m_0)\to\rho_\star(\cdot)$ for $p\in[1,\infty)$. Next we show that $\rho_\star(\cdot)$ solves (A.3). Take as test function in (A.1) the following function

$$\beta(t)F(u);\quad F(u)>0;\quad C\ge\beta(t)>\delta>0;\quad \beta'(t)\ge 0,\quad (u,t)\in\Lambda\times\mathbb{R}^+$$

$\beta\in C^2(R^+)$, $F\in C^2(\Lambda)$ vanishing at the boundary. Then for all $t>0$, see (A.1), we have

$$\begin{aligned}&\int_\Lambda du\big\{\beta(t)F(u)\rho(t,u)-\beta(0)F(u)\rho_0(u)\big\}-\int_0^t ds\beta'(s)\int_\Lambda duF(u)\rho(s,u)\\&\quad=\sum_{i,j}\int_0^t ds\beta(s)\bigg\{\int_\Lambda du\,A_{i,j}(\rho(s,u))\frac{\partial^2}{\partial_{i,j}}F(u)-\int_\Gamma A_{i,j}(b(u))\partial_{n_1}F(u)dS\bigg\}.\end{aligned}\tag{A.9}$$

Divide by t the left and right side of (A.9) and then let $t\to\infty$. For the left side we have

$$\frac{1}{t}\bigg\{\int_\Lambda du\big\{\beta(t)F(u)\rho(t,u)-\beta(0)F(u)\rho_0(u)\big\}-\int_0^t ds\beta'(s)\int_\Lambda duF(u)\rho(s,u)\bigg\}\to 0.\tag{A.10}$$

By continuity of $A(\cdot)$ and since by assumption $\lim_{s\to\infty}\beta(s)=\beta(\infty)>0$

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\sum_{i,j}\int_0^t ds\beta(s)\bigg\{\int_\Lambda du\,A_{i,j}(\rho(s,u))\frac{\partial^2}{\partial_{i,j}}F(u)-\int_\Gamma A_{i,j}(b(u))\partial_{n_1}F(u)dS\bigg\}\\&\quad=\beta(\infty)\sum_{i,j}\bigg\{\int_\Lambda du\,A_{i,j}(\rho_\star(u))\frac{\partial^2}{\partial_{i,j}}F(u)-\int_\Gamma A_{i,j}(b(u))\partial_{n_1}F(u)dS\bigg\}.\end{aligned}\tag{A.11}$$

By (A.10) we then obtain

$$\beta(\infty)\sum_{i,j}\left\{\int_{\Lambda} du\, A_{i,j}(\rho_\star(u))\frac{\partial^2}{\partial_{i,j}}F(u) - \int_{\Gamma} A_{i,j}(b(u))\partial_{n_1}F(u)dS\right\} = 0.$$

Therefore $\rho_\star$ is a solution of (A.3). The same can be argued for ρ^*. □

Proposition A.9 *There exists an unique weak solution of* (2.21).

Proof Existence of weak solution $\bar\rho$ of (2.21) is warranted by the tightness of the sequence Q_s^N, see Proposition 4.1. Arguing as in Theorem 5.2, p. 277 of [15] and comments p. 276 before Theorem 5.1 we deduce that $\max_{u\in\Lambda}|\nabla\bar\rho| \le M$ and $\bar\rho \in H^2(\Lambda)$. Further one can show, as in Theorem 6.1. p. 281 of [15], that $\bar\rho_{u_i}$, $i = 1,\dots,d$, are Holder continuous on Λ. The uniqueness is then a consequence of a comparison principle, see for example Lemma 10.7, p. 268 of [10]. □

Theorem A.10 (Global stability) *Let $D(\cdot)$ in* (2.15) *be Lipschitz. Let $\rho(t,\rho_0)$ be the solution of* (2.15) *with initial datum ρ_0, $0 \le \rho_0(u) \le 1$, $u \in \Lambda$, and $\bar\rho$ the stationary solution of* (2.15). *We have for all $p \ge 1$*

$$\lim_{t\to\infty}\int_{\Lambda}|\rho(t,u) - \bar\rho(u)|^p du = 0.$$

Proof Apply Lemma A.8 taking $m_0(u) = 0$ and $m_1(u) = 1$ for $u \in \Lambda$. By the uniqueness of the stationary solution of (2.15), see Proposition A.9, we deduce that $\rho^* = \rho_\star$ and the theorem is proved. □

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